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LIE ALGEBRA COHOMOLOGY AND THE GENERALIZED BOREL-WEIL THEOREM

By

Bertram Kostant

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Department of Mathematics
University of California
Berkeley 4, California

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Lie algebra cohomology and the generalized Borel-Weil Theorem

by Bertram Kostant

1. Introduction. 1. The present paper will be referred to as Part I. A subsequent paper entitled, "Lie algebra cohomology and generalized Schubert cells," will be referred to as Part II.

Let G be a complex semi-simple Lie group. Let $B \subseteq G$ be a Borel subgroup (a maximal connected solvable subgroup) and let Y be the generalized "flag" manifold G/B.

Now let M be the commutator subgroup of B so that M is a maximal unipotent subgroup of G. Let $\underline{g}$, $\underline{b}$, and $\underline{m}$ be, respectively, the Lie algebras of G, B and M.

Now let ν be an irreducible representation of $\underline{g}$ on a finite dimensional vector space V and let $\pi = \nu|\underline{m}$ be the restriction of ν to $\underline{m}$.

In [2] (see 15.3) Bott discovered the "strange" equality

$$\dim H^j(\underline{m},V) = \dim H^{2j}(Y,\underline{C}) \qquad (1.1.1)$$

where the common dimension of these cohomology groups equals the number of elements of the Weyl group of G which take exactly j positive roots into negative roots. Papers, Parts I and II had their origin in trying to "explain" this equality.

1.2. Let $\underline{a}$ be an arbitrary complex Lie algebra and let π be a representation of $\underline{a}$ on a finite dimensional vector space V.

In case $\underline{a}$ is semi-simple the nature of the cohomology groups $H^j(\underline{a},V)$ formed with respect to π are of course well known. However, in the general case very little is known about either the interpretation of $H^j(\underline{a},V)$ or how to compute it. For us attention was focused on just questions by the identity (1.1.1). In that case, of course, $\underline{a}$ is the Lie algebra $\underline{m}$ and π arises from a representation of $\underline{g}$. However, even here, Bott was able to obtain the left side of (1.1.1) only as a consequence of his generalized Borel-Weil theorem. This method (described by Bott as "obviously unsatisfactory") uses, besides representation theory, a number of deep results in algebraic geometry.

In this paper we introduce a technique (employing only representation theory) which not only yields $H(\underline{a},V)$, for a number of cases including the case above, but, (as it turns out) more significantly, how $H(\underline{a},V)$ transforms under the action of a certain group. Among other applications we then find that the generalized Borel-Weil itself follows easily from these results. But more than this the methods serve purposes other than the determination of cohomology groups. In fact in Part II they (in particular Theorem 5.7) play an important role in our extension of the Schubert calculus.

The main results of Part I are Theorems 4.4, 5.7, and 5.14. Applications are given in §§ 6, 7 and 8.

1.3. The following is a brief description of what is done in Part I. First, however, we wish to remark that the method of the Laplacian to determine cohomology is used in both Part I and

Part II. In Part I it arises in a conventional manner (from a coboundary operator on a cochain complex on which a positive definite Hermitian inner product has been defined). In Part II, however, there is no underlying Hermitian structure. None is actually required. The condition needed between a boundary operator and a coboundary operator to define a Laplacian, with the desired properties, we call disjointness. The details are stated in § 2.

In § 3, for one thing, we consider $H(\underline{a},V)$ under certain assumptions. First of all it is assumed that $\underline{a}$ is a subalgebra of a complex semi-simple Lie algebra $\underline{g}$. This, in itself, is no restriction since every Lie algebra can be so regarded. However, secondly, it is assumed that the representation π arises, as the restriction to $\underline{a}$, from a representation ν of $\underline{g}$. Let a real compact form $\underline{k}$ of $\underline{g}$ be fixed once and for all. We then observe, using the condition above on π, that the cochain complex $C(\underline{a},V)$ possesses a natural positive definite Hermitian structure. A Laplacian L_π can then be defined on $C(\underline{a},V)$ so that the determination of $H(\underline{a},V)$ becomes the problem of finding the kernel of L_π.

In order to find this kernel we seek an expression for L_π in terms of such computable operators as $\Theta(x)$ (Θ is the adjoint representation) and $\nu(x)$ where $x \in \underline{g}$. Proposition 3.13 is the main tool needed to prove Theorem 4.4 which succeeds in doing this when it is assumed that $\underline{a}$ is a Lie summand (see below).

In § 4 we introduce the notion of a Lie summand. Let $\underline{a}$ be a Lie subalgebra of $\underline{g}$. Let $\underline{a}^{o}$ be the set of all $x \in \underline{g}$ such that $(x,y) = 0$ for all $y \in \underline{a}$ where the bilinear form is the Cartan-Killing form. Then $\underline{a}$ is called a Lie summand if $\underline{a}^{o}$ is again a Lie subalgebra of $\underline{g}$.

The maximum nilpotent Lie subalgebra $\underline{m}$ is a Lie summand. But more than this, any Lie subalgebra $\underline{u}$ of $\underline{g}$ which contains $\underline{m}$ is a Lie summand. Theorem 4.4 gives the desired expression for L_{π} when $\underline{a}$ is a Lie summand.

In § 5, the main section, we are principally concerned with the case where $\underline{a}$ is the nilpotent Lie summand $\underline{n} = \underline{u}^{o}$ where $\underline{u}$ is an arbitrary Lie subalgebra containing $\underline{b}$. In such a case $\underline{n}$ is the maximal nilpotent ideal of $\underline{u}$. Furthermore $\underline{u}$ splits as an extension of $\underline{n}$ so that $\underline{u}$ can be written as the Lie algebra semi-direct sum $\underline{u} = \underline{g}_1 + \underline{n}$ where $\underline{g}_1$ is reductive in $\underline{g}$.

Theorem 5.7 gives the spectral resolution of L_{π} on $C(\underline{n},V)$. The cochain complex $C(\underline{n},V)$ is a representation space for $\underline{g}_1$ and we find that the eigenspaces of L_{π} correspond to the various irreducible representations of $\underline{g}_1$ and the eigenvalues are given in terms of the lengths of the corresponding highest weights. In Part I we are concerned only with $H(\underline{n},V)$ so that the non-zero eigenvalues are ignored here. However, in Part II Theorem 5.7 is needed because of what it says about certain non-zero eigenvalues.

Theorem 5.14 yields the cohomology group $H(\underline{n},V)$ and how it decomposes under the action of $\underline{g}_1$. The left side of (1.1.1)

is a special case of this result. To go from Theorem 5.7 to Theorem 5.14, techniques in representation theory are used. An important role here is played by the "spin" of the adjoint representation.

In § 6, as an application, the generalized Borel-Weil Theorem is proved. Needed for this is an auxiliary result of Bott (Theorem 1 in [2]). This result is actually relatively easy to prove. A proof considerably simpler than the one given in [2] is sketched here (see Remark 6.3).

Bott was the first one to observe that a proof of Weyl's character formula followed from a knowledge of $H(\underline{m},V)$. Such a proof is in fact quite interesting since it interprets the numerator and denominator in the formula to be "Euler characteristics". In § 7, we show how such a proof goes. In fact, more generally, we obtain a formula which works for disconnected groups.

Since every algebraic Lie algebra $\underline{u}$ decomposes into a semi-direct sum $\underline{u} = \underline{g}_1 + \underline{n}$, where $\underline{n}$ is a nilpotent ideal and $\underline{g}_1$ is reductive, the technique of § 7 seems particularly suited for generalizing Weyl's formula to algebraic groups.

In § 9 Theorem 5.14 is applied to the case representing the opposite extreme to one involved on the left side of (1.1.1). Namely to the case when n is commutative. Such a case arises in connection with complex symmetric spaces. In such a case Theorem 5.14 yields a generalization of a result of Ehresmann on how the holomorphic p-vectors at a point of the Grassmannian

(or its analogues for the orthogonal and symplectic groups) decompose under the action of the isotropy group at that point. We go into considerable details here since the results will be used in Part II.

1.4. Let $\underline{m}$ be as in §1.1. But now let $V = \underline{m}$ and let π be the adjoint representation of $\underline{m}$ on $\underline{m}$. Consider $H(\underline{m},\underline{m})$. Since $\underline{m}$ is not a $\underline{g}$ module the results of §5 do not apply here. Nevertheless we are able to slightly modify them so that at any rate $H^1(\underline{m},\underline{m})$ may be determined. But $H^1(\underline{m},\underline{m})$ is the Lie algebra of outer derivations of $\underline{m}$ modulo the inner ones. We are thus able to compute the full automorphism group of $\underline{m}$. The result and applications of it will be considered elsewhere.

2. Some definitions and notations. 1. Let C be a finite dimensional vector space over a field F. Let d and δ be linear operators on C such that $d^2 = \delta^2 = 0$. We will say that d and δ are disjoint if

$$d\delta x = 0 \quad \text{implies} \quad \delta x = 0$$

and

$$\delta dx = 0 \quad \text{implies} \quad dx = 0$$

for all $x \in C$. In such a case we define an operator L, referred to as a Laplacian, by putting

$$L = d\delta + \delta d \tag{2.1.1}$$

and note, after the next definition, the following proposition.

In any operator A on C let Ker A and Im A be, respectively, the kernel and range of A.

Prop. 2.1. Let the notation be as above. Assume d and δ are disjoint and let L be defined by (2.1.1). Then

$$\text{Ker } L = \text{Ker } d \cap \text{Ker } \delta . \tag{2.1.2}$$

Also one has a direct sum (a "Hodge decomposition"),

$$C = \text{Im } d + \text{Im } \delta + \text{Ker } L \tag{2.1.3}$$

so that if the derived space Ker d/Im d of d is denoted by H(C) and

$$Q : \text{Ker } d \longrightarrow H(C)$$

is the canonical mapping then

$$Q : \text{Ker } L \longrightarrow H(C) \tag{2.1.4}$$

is a bijection.

Proof. Statement (2.1.4) is an immediate consequence of (2.1.2) and (2.1.3) and the definition of disjointness for d and δ. But statement (2.1.3) is an immediate consequence of the observation that Im L $\subseteq$ Im d + Im δ, (2.1.2) which implies that Im d + Im δ) $\cap$ Ker L = 0 and the fact that dim Ker L + dim Im L = dim C. It suffices therefore to prove only (2.1.2) or that Lx = 0 implies dx = δ x = 0. Assume Lx = 0. Put y = - δ dx. Then δ y = 0 and also y = dδ x. Thus δ d(δ x) = 0. But by disjointness this implies d δ x = 0 which, for the same reason, implies δ x = 0. Similarly dx = 0. QED.

Remark 2.1. For later use we record the observation, made implicitly in the proof above, that Im L $\cap$ Ker L = 0 and in fact

$$\text{Im } L = \text{Im } d + \text{Im } \delta \ . \qquad (2.1.5)$$

When C is a cochain complex and d is the coboundary operator the elements of Ker L will often be called harmonic cocycles. In such a case Proposition 2.1 asserts that every cocycle is cohomologous to one and only one harmonic cocycle.

2.2. Throughout the paper the following conventions have been adopted. We denote by End C the algebra of all linear operators on C and by C' the dual space to C. Also we denote with pointed brackets $\langle x, f \rangle$ the value the function of bilinearity between C and C', takes on $x \in C$ and $f \in C'$.

If a symmetric bilinear form (resp. Hermitian inner product, i.e. Hermitian structure, assuming $\underline{F} = \underline{C}$, the field of complex numbers) is defined on C we will denote with round

brackets (x,y) (resp. with curly brackets $\{x,y\}$ the value the form (resp. inner product) takes on $x,y \in C$. When there is no danger of confusion the form {resp. inner product} itself will be denoted by (C) (resp. $\{C\}$). In case (C) (resp. $\{C\}$) is non-singular (resp. positive definite) as will always be the case in this paper and $A \in \text{End } C$ we denote by A^t (resp. A^*, the adjoint of A) the operator on C defined by

$$(Ax,y) = (x,A^t y) \qquad (2.2.1)$$

(resp. $\{Ax,y\} = \{x,A^*y\}$)

Remark 2.2. If C is a vector space over $\underline{C}$ the words orthogonal, orthonormal and orthocomplement will always be understood to be with respect to a positive definite Hermitian structure $\{C\}$ which has been defined on C and not with respect to a bilinear form (C) which may also have been defined on C.

2.3. It will be assumed from this point on that, unless statements are made to the contrary, every vector space considered in this paper is over $\underline{C}$ and that every homomorphism of one vector space into another is $\underline{C}$-linear. More generally every homomorphism of one complex Lie group into another will be assumed to be holomorphic so that in particular representations of such groups are understood to be holomorphic.

Remark 2.3. Assume $\{C\}$ is a positive definite Hermitian structure on C. Let d be an operator on C such that $d^2 = 0$ and let d^* be the adjoint of d with respect to $\{C\}$. Obviously $d^{*2} = 0$. But we observe also that since $\{C\}$ is positive

definite d and d* are disjoint. Furthermore the Laplacian $L = dd^* + d^*d$ is self-adjoint and the decomposition (2.1.3) for $\delta = d^*$ is an orthogonal direct sum decomposition.

3. Cochain complexes defined by Lie algebras and Hermitian Structures. 1. Let $\underline{a}$ be a complex Lie algebra. Then the exterior algebra $\Lambda\,\underline{a}$ over $\underline{a}$ together with the boundary operator ∂ on $\Lambda\,\underline{a}$ given by

$$\partial(x_1 \wedge \ldots \wedge x_k) = \sum_{i<j} (-1)^{i+j+1}[x_i, x_j] \wedge x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots \wedge x_k, \tag{3.1.1}$$

where $x_i \in \underline{a}$, is a chain complex which one denotes by $C_*(\underline{a})$. The derived space of homology is denoted by $H_*(\underline{a})$. Covariantly the exterior algebra $\Lambda\,\underline{a}'$ over the dual $\underline{a}'$ to $\underline{a}$ is canonically identified with the dual to $\Lambda\,\underline{a}$ and $\Lambda\,\underline{a}'$ together with the coboundary operator d, defined as the negative transpose of ∂, is a cochain complex which one denotes by $C(\underline{a})$. The derived space of cohomology (for the present we ignore its ring structure) is denoted by $H(\underline{a})$. (See [7]).

More generally let V be a vector space and let

$$\pi : \underline{a} \longrightarrow \text{End } V$$

be a representation of $\underline{a}$ on V. Let d_1 be the operator on the tensor product $\Lambda\,\underline{a}' \otimes V$ defined by putting $d_1 = d \otimes 1$, and also, regarding $\Lambda\,\underline{a}' \otimes V$ as the space of all linear maps p from $\Lambda\,\underline{a}$ to V, let d_2 be the operator on $\Lambda\,\underline{a}' \otimes V$ given by

$$d_2 p(x_1 \wedge \ldots \wedge x_k) = \sum_{i=1}^{k} (-1)^{i+1} \pi(x_i) p(x_1 \ldots \wedge \hat{x}_i \ldots \wedge x_k) \tag{3.1.2}$$

Now define $d_\pi = d_1 + d_2$. Then $d_\pi^2 = 0$ and the space $\Lambda \underline{a}' \otimes V$ together with the coboundary operator d_π is a cochain complex which one designates by $C(\underline{a},V)$. The derived space of cohomology is denoted by $H(\underline{a},V)$.

3.2. Now let $\underline{g}$ be a complex semi-simple Lie algebra and let $(\underline{g})$ be the Cartan-Killing form on $\underline{g}$. The form $(\underline{g})$ induces an isomorphism of $\underline{g}$ onto $\underline{g}'$ which extends to an algebra isomorphism of $\Lambda \underline{g}$ onto its dual $\Lambda \underline{g}'$. The latter in turn induces a non-singular symmetric bilinear form $(\Lambda \underline{g})$ on $\Lambda \underline{g}$ which, explicitly, is given by

$$(x_1 \wedge \ldots \wedge x_p,\ y_1 \wedge \ldots \wedge y_q) = 0 \text{ if } p \neq q \qquad (3.2.1)$$
$$= \det (x_i, y_j) \text{ if } p = q$$

3.3. A real Lie subalgebra $\underline{k}$ of $\underline{g}$ is called a compact form of $\underline{g}$ if (1) $\underline{g} = \underline{k} + i\underline{k}$ is a real direct sum and (2) the bilinear form $(\underline{g})$ is negative definite on $\underline{k}$.

A compact form of $\underline{g}$ denoted by $\underline{k}$ is henceforth assumed to be fixed once and for all. Let $\underline{q} = i\underline{k}$. Then $\underline{q}$ is a real subspace of $\underline{g}$ on which $(\underline{g})$ is positive definite. Where $\underline{R}$ denotes the real field let $\Lambda_{\underline{R}}\underline{q}$ be the subalgebra of $\Lambda \underline{g}$ generated over $\underline{R}$ by $\underline{q}$ and $\underline{R}$. Clearly $(\Lambda \underline{g})$ is positive definite on $\Lambda_{\underline{R}}\underline{q}$ and

$$\Lambda \underline{g} = \Lambda_{\underline{R}}\underline{q} + i\Lambda_{\underline{R}}\underline{q}$$

is a real direct sum.

A *-operation is now introduced into $\Lambda \underline{g}$ by defining

$$(u + iv)^* = u - iv$$

for all $u,v \in \Lambda_{\underline{R}}\underline{q}$. It follows easily that this operation is a conjugate linear automorphism of $\Lambda \underline{g}$.

Since

$$(u^*,v^*) = (\overline{u,v})$$

for every $u,v \in \Lambda \underline{g}$ we can define a Hermitian inner product $\{\Lambda \underline{g}\}$ on $\Lambda \underline{g}$ by putting

$$\{u,v\} = (u,v^*) \tag{3.3.1}$$

for all $u,v \in \Lambda \underline{g}$. Since $(\Lambda \underline{g})$ is positive definite on $\Lambda_{\underline{R}}\underline{q}$ it follows immediately that $\{\Lambda \underline{g}\}$ is positive definite on $\Lambda \underline{g}$.

Let $A \in$ End $\Lambda \underline{g}$ and let A^t, $A^* \in$ End $\Lambda \underline{g}$ be defined as in § 2.2. It follows immediately from (3.3.1) that A^t and A^* are related by

$$A^*u = (A^t(u^*))^* \tag{3.3.2}$$

for every $u \in \Lambda \underline{g}$. Substituting A^t for A in (3.3.2) and then A^* for A and u^* for u it follows at once that

$$A^{t*} = A^{*t} \tag{3.3.3}$$

3.4. For any subspace $\underline{b} \subseteq \Lambda \underline{g}$ let

$$\underline{b}^* = \{u^* \in \Lambda \underline{g} \mid u \in \underline{b}\} .$$

Obviously $\underline{b}^*$ is again a (complex) subspace of $\underline{g}$.

We now assume that the arbitrary Lie algebra $\underline{a}$ of § 2.3 is a Lie subalgebra of $\underline{g}$. Since the $*$-operation is a conjugate linear automorphism of $\Lambda \underline{g}$ it is obvious that

$$(\Lambda \underline{a})^* = \Lambda \underline{a}^* . \tag{3.4.1}$$

We now define a degree preserving linear mapping

$$\tilde{\eta}: \Lambda \underline{a}^* \longrightarrow \Lambda \underline{a}'$$

by the relation

$$\langle u, \tilde{\eta}(v) \rangle = (u,v) \qquad (3.4.2)$$

for $u \in \Lambda \underline{a}$, $v \in \Lambda \underline{a}^*$. It is obvious that $\tilde{\eta}$ is a homomorphism.

Lemma 3.4. The mapping $\tilde{\eta}$ is an algebra isomorphism (onto).

Proof. By dimension it suffices only to show that $\tilde{\eta}$ is an injection. Let $v \in \Lambda \underline{a}^*$. Then by (3.4.1) $v = u^*$ for some $u \in \Lambda \underline{a}$. But

$$\begin{aligned} \langle u, \tilde{\eta}(v) \rangle &= (u,v) \\ &= (u,u^*) \\ &= \{u,u\}. \end{aligned}$$

Thus $\tilde{\eta}(v) = 0$ implies $u = 0$ since $\{\Lambda \underline{g}\}$ is positive definite But this implies $v = 0$. QED.

Now let

$$\eta : \Lambda \underline{a}' \longrightarrow \Lambda \underline{g} \qquad (3.4.3)$$

be the monomorphism defined so that $\eta \tilde{\eta}$ is the identity on $\Lambda \underline{a}^*$. It is obvious that $\Lambda \underline{a}^*$ is the image of η.

We now define a positive definite Hermitian structure $\{\Lambda \underline{a}'\}$ on $\Lambda \underline{a}'$ by the relation

$$\{f,g\} = \{\eta(f), \eta(g)\} \ . \qquad (3.4.4)$$

for any $f,g \in \Lambda \underline{a}'$.

3.5. Now let

$$\nu : \underline{g} \longrightarrow \text{End } V$$

be a representation of $\underline{g}$ on a finite dimensional vector space V. Then one knows that there exists a positive definite Hermitian structure $\{V\}$ on V (which is unique up to positive multiple if ν is irreducible) such that

$$(\nu(z))^* = \nu(z^*) \tag{3.5.1}$$

for any $z \in \underline{g}$. (This is, a $\{V\}$ exists such that all the operators in $\nu(k)$ are skew-Hermitian.)

But now the Hermitian structure $\{\Lambda \underline{a}'\}$ and $\{V\}$ induce a positive definite Hermitian structure $\{\Lambda \underline{a}' \otimes V\}$ on $\Lambda \underline{a}' \otimes V$ by the relation

$$\{f \otimes s, g \otimes t\} = \{f,g\}\{s,t\}$$

where $f,g \in \Lambda \underline{a}'$ and $s,t \in V$. In a similar way a positive definite Hermitian structure $\{\Lambda \underline{g} \otimes V\}$ on $\Lambda \underline{g} \otimes V$ is induced by $\{\Lambda \underline{g}\}$ and $\{V\}$. It is clear then of course that

$$\eta \otimes 1 : \Lambda \underline{a}' \otimes V \longrightarrow \Lambda \underline{g} \otimes V \tag{3.5.2}$$

is an isometry onto the subspace $\Lambda \underline{a}^* \otimes V$ of $\Lambda \underline{g} \otimes V$.

Let $\pi = \nu|\underline{a}$ be the restriction of ν to $\underline{a}$. Now form the cochain complex $C(\underline{a},V)$ with respect to π as in § 3.1 and let

$$Q : \operatorname{Ker} d_\pi \longrightarrow H(\underline{a},V)$$

be the canonical map of the space of cocycles onto the space of cohomology. Let d_π^* be the adjoint of d_π with respect to $\{\Lambda \underline{a}' \otimes V\}$. Then as noted in Remark 2.3 d_π and d_π^* are disjoint and if we let

$$L_\pi = d_\pi d_\pi^* + d_\pi^* d_\pi \tag{3.5.3}$$

be the corresponding Laplacian it is well known or by Prop. 2.1 it follows that Ker $L_\pi \subseteq$ Ker d_π and

$$Q : \text{Ker } L_\pi \longrightarrow H(\underline{a},V) \qquad (3.5.4)$$

is a bijection. To determine $H(\underline{a},V)$ it is enough therefore to determine Ker L_π.

3.6. An operator $A \in \text{End} \bigwedge \underline{g}$ is said to be a derivation of degree j if (1) A maps $\bigwedge^i \underline{g}$ into $\bigwedge^{i+j} \underline{g}$ for all i and (2)

$$A(u \wedge v) = A(u) \wedge V + (-1)^{ij} u \wedge A(v)$$

if $u \in \bigwedge^i \underline{g}$ and $v \in \bigwedge \underline{g}$.

For any $u \in \bigwedge \underline{g}$ let $\epsilon(u) \in \text{End} \bigwedge \underline{g}$ be the operator define by $\epsilon(u)v = u \wedge v$ for all $v \in \bigwedge \underline{g}$ (left exterior multiplication by u). If $u \in \bigwedge^i \underline{g}$ and A is a derivation of degree j then it is straightforward to verify that

$$\epsilon(u)A \text{ is a derivation of degree } i+j \qquad (3.6.1)$$

Now for any $u \in \bigwedge \underline{g}$ the operator $\iota(u) \in \text{End} \bigwedge \underline{g}$ of left interior multiplication by u is defined by putting

$$\iota(u) = (\epsilon(u))^t \qquad (3.6.2)$$

(To avoid confusion it should be remarked that in the usual notation $\iota(u)$ operates on $\bigwedge \underline{g}'$ and the operators of interior multiplication on $\bigwedge \underline{g}$ are of the form $\iota(f)$ where $f \in \bigwedge \underline{g}'$. However the bilinear form $(\bigwedge \underline{g})$ permits us to regard $\iota(u)$ as an operator on $\bigwedge \underline{g}$ and to ignore $\bigwedge \underline{g}'$).

If $z \in \bigwedge^1 \underline{g}$ then one knows that $\iota(z)$ is a derivation of degree -1 on $\bigwedge \underline{g}$. In fact by (3.2.1)

$$\iota(z)\ y_1\wedge\ldots\wedge y_k = \sum_{i=1}^{k} (-1)^{i+1}(z,y_i)\ y_1\ \ldots\wedge\hat{y}_i\ \ldots\wedge y_k \quad (3.6.3)$$

Now if A is a derivation of degree j it is obvious that A vanishes on $\Lambda^0\underline{g}$ and is uniquely determined by its restriction to $\Lambda^1\underline{g}$. Conversely given any linear transformation A_0: $\Lambda^1\underline{g} \longrightarrow \Lambda^{j+1}\underline{g}$ there exists a derivation of degree (necessarily unique) which extends A_0. In fact if y_i and z_j are dual bases of $\underline{g}$ with respect to $(\underline{g})$ then by (3.6.1)

$$A = \sum_i \epsilon(A_0 y_i)\,\iota(z_i)$$

is clearly such a derivation.

3.7. Let $\underline{r}$ be any subspace of $\underline{g}$. Denote by $r^\perp$ its orthocomplement in $\underline{g}$ and by $\Gamma_{\underline{r}} \in \mathrm{End}\,\Lambda\underline{g}$.

$$\Gamma_r : \Lambda\underline{g} \longrightarrow \Lambda\underline{r}$$

the orthogonal (Hermitian) projection of $\Lambda\underline{g}$ onto $\Lambda\underline{r}$. If $\underline{p} \subseteq \underline{g}$ is any subspace denote by $I(\underline{p})$ the ideal in $\Lambda\underline{g}$ generated by $\underline{p}$. We now observe that since

$$\Lambda\underline{g} = \Lambda\underline{r} + I(\underline{r}^\perp)$$

is an orthogonal direct sum it follows that $\Gamma_{\underline{r}}$ is a homomorphism. This is clear since $\Gamma_{\underline{r}}$ is a homomorphism on $\Lambda\underline{r}$ and its kernel, $I(\underline{r}^\perp)$, is an ideal.

Next we observe that

$$\Gamma_{\underline{r}}^t = \Gamma_{r^*} \quad (3.7.1)$$

Indeed since $\Gamma_{\underline{r}}$ is a Hermitian projection it follows from

(3.3.3) that $\Gamma^t_{\underline{r}}$ is also a Hermitian projection. On the other hand it follows from (3.3.2) that the range of $\Gamma^t_{\underline{r}}$ is $(\Lambda \underline{r})^* = \Lambda \underline{r}^*$. This proves (3.7.1).

3.8. Now let $\gamma \in \mathrm{End}\,\Lambda \underline{g}$ be the boundary operator for the chain complex $C_*(\underline{g})$. It is obvious that $\Lambda \underline{g}$ is stable under γ and that

$$\gamma \mid \Lambda \underline{a} = \partial \tag{3.8.1}$$

where, as in § 3.1, $\partial \in \mathrm{End}\,\Lambda \underline{a}$ is the boundary operator for the chain complex $C_*(\underline{a})$.

Now define an operator $c \in \mathrm{End}\,\Lambda \underline{g}$ by setting

$$c = -\gamma^t \tag{3.8.2}$$

An immediate consequence of (3.7.1) is the following lemma which asserts that the coboundary operator $d \in \mathrm{End}\,\Lambda \underline{a}'$ for the chain complex $C(\underline{a})$ corresponds under η to the operator $\Gamma_{\underline{a}^*} c$ on $\Lambda \underline{a}^*$.

Lemma 3.8. *Let $\underline{a}$ be a Lie subalgebra of $\underline{g}$. Let $f \in \Lambda \underline{a}'$. Then*

$$\eta(df) = \Gamma_{\underline{a}^*} c(\eta f)$$

Proof. It clearly suffices, by Lemma 3.4, to show that

$$(u, \eta(df)) = (u, \Gamma_{\underline{a}^*} c(\eta f))$$

for all $u \in \Lambda \underline{a}$. But by definition of η one has $(u, \eta\, df) = \langle u, df \rangle = -\langle \partial u, f \rangle$. On the other hand $(u, \Gamma_{\underline{a}^*} c(\eta f)) = (\Gamma_{\underline{a}} u, c(\eta f)) = (u, c(\eta f))$ by (3.7.1). But $(u, c(\eta f)) = -(\gamma u, \eta f) = -(\partial u, \eta f)$ by (3.8.1) and $-(\partial u, \eta f) = -\langle \partial u, f \rangle$. This proves the lemma. QED.

Remark 3.8. Recall that η is an algebra monomorphism. Since, as one knows, d is a derivation of degree 1 on $\Lambda \underline{a}'$ it follows then from Lemma 3.8 that the restriction of $\Gamma_{\underline{a}^*}c$ to $\Lambda \underline{a}^*$ is a derivation of degree 1 of $\Lambda \underline{a}^*$. Substituting $\underline{g}$ for $\underline{a}$ it then follows that c is a derivation of degree 1 of $\Lambda \underline{g}$. But this fact implies, more generally, that the restriction of $\Gamma_{\underline{r}^*}c$ to $\Lambda \underline{r}^*$, for any subspace $\underline{r} \subseteq \underline{g}$ is a derivation of $\Lambda \underline{r}^*$. This is clear since, as has been observed (see § 3.7), $\Gamma_{\underline{r}^*}$ is a homomorphism of $\Lambda \underline{g}$.

3.9. Now since $\underline{k}$ is a real Lie subalgebra of $\underline{g}$ this implies (see § 3.3) that $[\underline{q},\underline{q}] \subseteq i\underline{q}$. But then recalling the definition of γ it follows that

$$\gamma : \Lambda_{\underline{R}}\underline{q} \longrightarrow i\Lambda_{\underline{R}}\underline{q}$$

This, however, implies, from the definition of the *-operation, that γ anti-commutes with the *-operation. That is, for any $u \in \Lambda \underline{g}$

$$\gamma u = (-\gamma(u^*))^*. \tag{3.9.1}$$

Remark 3.9. Note that (3.9.1) implies $\underline{a}^*$ is a Lie subalgebra of $\underline{g}$ if and only if $\underline{a}$ is a Lie subalgebra of $\underline{g}$. In case $\underline{a}$ is a Lie subalgebra and $\underline{a} = \underline{a}^*$ it follows that $\underline{a}$ is necessarily reductive in $\underline{g}$ since it clearly arises as the complexification of a Lie subalgebra of $\underline{k}$.

Now by definition $-\gamma = c^t$. Applying (3.3.2) to (3.9.1) it then follows that

$$c^* = \gamma \ . \tag{3.9.2}$$

We recall that the *-operation is a conjugate linear automorphism of $\Lambda \underline{g}$. It follows then that for any $u, v \in \Lambda \underline{g}$

$$\epsilon(u^*)v = (\epsilon(u)v^*)^* .$$

But $\epsilon(u) = \iota(u)^t$. Applying (3.3.2) once more it follows that

$$\iota(u)^* = \epsilon(u^*) \tag{3.9.3}$$

Now let

$$\theta : \underline{g} \longrightarrow \mathrm{End}\, \Lambda \underline{g}$$

denote the adjoint representation of $\underline{g}$ on $\Lambda \underline{g}$. Thus $\theta(y)$ for every $y \in \underline{g}$, is the unique derivation of degree 0 on $\Lambda \underline{g}$ which on $\underline{g}$ satisfies $\theta(y)z = [y,z]$.

Since $(\underline{g})$ is invariant under θ it is clear that for every $y \in \underline{g}$

$$(\theta(y))^t = -\theta(y) \tag{3.9.4}$$

But now it is well known (when considered on $\Lambda \underline{g}'$) that

$$\iota(y)c + c\,\iota(y) = \theta(y) \tag{3.9.5}$$

In fact since the left side of (3.9.5) is easily seen to be a derivation of degree 0 on $\Lambda \underline{g}$ it suffices to verify (3.9.5) on $\Lambda^1 \underline{g}$. But then (3.9.5) is an immediate consequence of the definition of c.

Now applying the operation $A \longrightarrow A^t$, $A \in \mathrm{End}\, \Lambda \underline{g}$, to (3.9.5) and taking negatives one also has by (3.6.2), (3.8.2) and (3.9.4) that

$$\epsilon(y)\gamma + \gamma\,\epsilon(y) = \theta(y) \tag{3.9.6}$$

On the other hand by taking the adjoint of (3.9.5) one gets,

by (3.9.2) and (3.9.3), the same expression as (3.9.6) on the left except that y* replaces y. It follows then that

$$(\theta(y))^* = \theta(y^*) \tag{3.9.7}$$

for any $y \in \underline{g}$.

3.10. Let $\underline{p}$ and $\underline{r}$ be two subspaces of $\underline{g}$. We now observe that there exists a linear mapping

$$\gamma_{\underline{p},\underline{r}}: \Lambda^2\underline{g} \longrightarrow \Lambda^1\underline{g}$$

such that for any $y, z \in \underline{g}$

$$\gamma_{\underline{p},\underline{r}}(y \wedge z) = \tfrac{1}{2}([\Gamma_{\underline{p}}y, \Gamma_{\underline{r}}z] - [\Gamma_{\underline{p}}z, \Gamma_{\underline{r}}y]) \tag{3.10.1}$$

This is clear since the right side of (3.10.1) is alternating in y and z. But now by § 3.6 there then exists a unique derivation $c_{\underline{p},\underline{r}}$ of degree 1 on $\Lambda\underline{g}$ such that for any $u \in \Lambda^2\underline{g}$ and $z \in \Lambda^1 z$

$$(c_{\underline{p},\underline{r}}z, u) = -(z, \gamma_{\underline{p},\underline{r}}u) \tag{3.10.2}$$

Although the right side of (3.10.1) is alternating in y and z we now make the observation that it is symmetric in $\underline{p}$ and $\underline{r}$ Consequently one has

$$c_{\underline{p},\underline{r}} = c_{\underline{r},\underline{p}} \tag{3.10.3}$$

Obviously if $\underline{r}_1$ and $\underline{r}_2$ are orthogonal subspaces of $\underline{g}$ one has

$$c_{\underline{p},\underline{r}_1} + c_{\underline{p},\underline{r}_2} = c_{\underline{p},\underline{r}_1+\underline{r}_2} \tag{3.10.4}$$

Let r be a subspace of $\underline{g}$. We now observe that

$$c_{\underline{r},\underline{r}} z = \Gamma_{\underline{r}^*} c\, z \tag{3.10.5}$$

for any $z \in \underline{g}$. Indeed since $\Gamma_{\underline{r}}$ is a homomorphism (see §3.7) it follows immediately that $\gamma_{\underline{r},\underline{r}} = \gamma\Gamma_{\underline{r}}$ on $\Lambda^2\underline{g}$. But then (3.10.5) follows at once from the fact that $-(\gamma\Gamma_{\underline{r}})^t = \Gamma_{\underline{r}^*}c$ (see (3.7.1)).

If $\underline{r} = \underline{a}$ where $\underline{a}$ is a Lie subalgebra of $\underline{g}$ it is clear from Lemma 3.8 and (3.10.5) that $c_{\underline{r},\underline{r}}$ will be significant in computing $H(\underline{a},V)$. It would therefore be convenient if one could express $c_{\underline{r},\underline{r}}$ in terms of such computable operations as exterior multiplication and the adjoint representation. This seems to be unlikely in the case of a general Lie subalgebra $\underline{a}$. On the other hand if $\underline{r}$ is any subspace of $\underline{g}$ one can find, as will soon be shown, just such an expression for the derivation $d_{\underline{r}}$ of $\Lambda\,\underline{g}$ defined by putting

$$d_{\underline{r}} = c_{\underline{r},\underline{r}} - c_{\underline{r}^\perp,\underline{r}^\perp} \tag{3.10.6}$$

But the point is that under certain assumptions (which are satisfied for the cases which interest us) $d_{\underline{r}}$ is a satisfactory replacement for $c_{\underline{r},\underline{r}}$. This is seen in comparing (3.10.5) and

Lemma 3.10. *Let $\underline{r}$ be a subspace of $\underline{g}$. Assume $\underline{r}^\perp$ is a Lie subalgebra of $\underline{g}$. Then for any $u \in \Lambda\,\underline{r}$.*

$$d_{\underline{r}} u = \Gamma_{\underline{r}^*} cu$$

Proof. Let $z \in \underline{r}^*$. We first observe that $c_{\underline{r}^\perp,\underline{r}^\perp} z = 0$.

For this it suffices to show that $(v, c_{\underline{r},\underline{r}}z) = 0$ for any $v \in \Lambda^2\underline{g}$. But, since $\underline{r}^\perp$ is a Lie subalgebra of $\underline{g}$, it is obvious that $\gamma_{\underline{r}^\perp,\underline{r}^\perp}$ maps $\Lambda^2\underline{g}$ into $\underline{r}^\perp$. Therefore since $z^* \in \underline{r}$,

$$\begin{aligned}(v, c_{\underline{r}^\perp,\underline{r}^\perp}z) &= -(\gamma_{\underline{r}^\perp,\underline{r}^\perp}v, z) \\ &= -\{\gamma_{\underline{r}^\perp,\underline{r}^\perp}v, z^*\} \\ &= 0\end{aligned}$$

Thus for any $z \in \underline{r}^*$ one has $d_{\underline{r}}z = c_{\underline{r},\underline{r}}z$. But then by (3.10.5) $d_{\underline{r}}$ and $\Gamma_{\underline{r}^*}c$ agree on $\Lambda^1\underline{r}^*$. On the other hand since the restriction of $\Gamma_{\underline{r}^*}c$ to $\Lambda\underline{r}^*$ is a derivation of $\Lambda\underline{r}^*$. (See Remark 3.8; that is, because $\Gamma_{\underline{r}^*}$ is a homomorphism of $\Lambda\underline{g}$) it follows that $d_{\underline{r}}$ and $\Gamma_{\underline{r}^*}c$ agree on $\underline{r}^*$. QED.

3.11. Let $\underline{r}$ be a subspace of $\underline{g}$. Let z_i, $1 \leq i \leq m$, be an orthonormal basis of $\underline{r}$. Let $y \in \underline{g}$. Writing (y, z_i^*) for $\{y, z_i\}$ it is obvious then that

$$\Gamma_{\underline{r}}y = \sum_{i=1}^{m} (y, z_i^*)z_i \qquad (3.11.1)$$

Now let $c_{\underline{r}} = c_{\underline{r},\underline{g}}$. To obtain the desired expression for $d_{\underline{r}}$ we first observe that $c_{\underline{r}}$ may be given by the following simple expression.

Lemma 3.11. *Let $\underline{r}$ be any subspace of $\underline{g}$. Then if z_i, $1 \leq i \leq m$, is an orthonormal basis of $\underline{r}$*

$$c_{\underline{r}} = \frac{1}{2} \sum_{i=1}^{m} \epsilon(z_i^*)\theta(z_i) \quad .$$

Proof. Since the right side of (3.11.2) is a derivation (see (3.6.1)) it suffices to verify the equality for elements in $\Lambda^1 \underline{g}$. Thus if $x,y,z \in \underline{g}$ it is enough to prove the equality

$$(\gamma_{\underline{r},\underline{g}}\, x\wedge y, z) = -\frac{1}{2}\,(x\wedge y, \sum_{i=1}^{m} \epsilon(z_i^*)\theta(z_i)z),$$

that is, to prove the equality (writing $y\wedge x = -x\wedge y$)

$$([\Gamma_{\underline{r}}x,y],z) - ([\Gamma_{\underline{r}}y,x],z) = \sum_{i=1}^{m} (y\wedge x,\ z_i^*\wedge[z_i,z]) \quad (3.11.2)$$

But by (3.2.1) where $p = q = 2$

$$(y\wedge x,\ z_i^*\wedge[z_i,z]) = (y,z_i^*)(x,[z_i,z]) - (x,z_i^*)(y,[z_i,z])$$

$$= (x,z_i^*)([z_i,y],z) - (y,z_i^*)([z_i,x],z)$$

since $\theta(z_i)^t = -\theta(z_i)$. Summing over i the equality in (3.11.2) follows immediately from (3.11.1). QED.

We now observe that

$$c_{\underline{r},\underline{r}} - c_{\underline{r}^\perp,\underline{r}^\perp} = c_{\underline{r}} - c_{\underline{r}^\perp} \quad (3.11.3)$$

In fact by (3.10.4) $c_{\underline{r}} = c_{\underline{r},\underline{r}} + c_{\underline{r},\underline{r}^\perp}$ and $c_{\underline{r}^\perp} = c_{\underline{r}^\perp,\underline{r}^\perp} + c_{\underline{r}^\perp,\underline{r}}$. But then (3.11.3) follows from (3.10.3) when p is replaced by $\underline{r}^\perp$. Now recalling the definition of $d_{\underline{r}}$ (3.10.6) the proof of the following proposition follows from Lemma 3.11 and (3.11.3).

Proposition 3.11. *Let $\underline{r}$ be any subspace of $\underline{g}$. Let $d_{\underline{r}}$ be the derivation of degree 1 of $\Lambda\,\underline{g}$ defined by (3.10.6). Let z_i, $1 \le i \le n$, be an orthonormal basis of $\underline{g}$ such that z_i for $i \le m$ is a basis of $\underline{r}$. Then*

$$d_{\underline{r}} = \frac{1}{2}\left(\sum_{i=1}^{m} \epsilon(z_i^*)\theta(z_i) - \sum_{j=m+1}^{n} \epsilon(z_j^*)\theta(z_j)\right)$$

As an analogy with the definition d_1 in §3.1 let $d_{\underline{r},1}$ be the operator on $\Lambda \underline{g} \otimes V$ defined by putting

$$d_{\underline{r},1} = d_{\underline{r}} \otimes 1 \tag{3.11.4}$$

3.12. Let $\underline{r}$ be a subspace of $\underline{g}$ and let z_i, $1 \leq i \leq m$, be an orthonormal basis of $\underline{r}$. We define an operator $d_{\underline{r},2}$ on $\Lambda \underline{g} \otimes V$ by putting

$$d_{\underline{r},2} = \sum_{i=1}^{m} \epsilon(z_i^*) \otimes \nu(z_i) \tag{3.12.1}$$

It is straightforward to verify that the definition is independent of the orthonormal basis chosen. Put

$$c_2 = d_{\underline{g},2} \tag{3.12.2}$$

Obviously $c_2 = d_{\underline{r},2} + d_{\underline{r}^\perp,2}$. Note then that we can write (and for "compatibility" with the expression for $d_{\underline{r},1}$ given by Proposition 3.11 and (3.11.4) it is convenient to do so)

$$d_{\underline{r},2} = \frac{1}{2}(c_2 + d_{\underline{r},2} - d_{\underline{r}^\perp,2}) \tag{3.12.3}$$

Now let $\underline{r} = \underline{a}$ be a Lie subalgebra of $\underline{g}$. Let $z_i' \in \underline{a}'$ be the basis of $\underline{a}'$ dual to the basis z_i of $\underline{a}$. It is then a simple matter to verify that the operator d_2 on $\Lambda \underline{a}' \otimes V$ defined by (3.1.2) may be given by

$$d_2 = \sum_{i=1}^{m} \epsilon(z_i') \otimes \nu(z_i)$$

where $\epsilon(z_i')$ is left exterior multiplication on $\Lambda \underline{a}'$ by z_i'. On the other hand since

$$(z_i, z_j^*) = \delta_{ij}$$

and since $z_j^* \in \underline{a}^*$ it follows from the definition of η that $\eta(z_i^*) = z_i^*$. But then for any $p \in \Lambda \underline{a}' \otimes V$ one obtains the relation

$$\eta \otimes 1(d_2 p) = d_{\underline{a},2}(\eta \otimes 1(p)) \tag{3.12.4}$$

3.13. Now let

$$\theta_\nu : \underline{g} \longrightarrow \mathrm{End}(\Lambda \underline{g} \otimes V)$$

be the representation of $\underline{g}$ in $\Lambda \underline{g} \otimes V$ formed by taking the tensor product of θ and ν. Thus for any $z \in \underline{g}$

$$\theta_\nu(z) = \theta(z) \otimes 1 + 1 \otimes \nu(z)$$

Now let $\underline{r} = \underline{a}$ be a Lie subalgebra of $\underline{g}$. Define

$$d_{\underline{a},\nu} = d_{\underline{a},1} + d_{\underline{a},2} \tag{3.13.1}$$

where $d_{\underline{a},1}$ and $d_{\underline{a},2}$ are given by (3.11.4) and (3.12.1).

But then by Proposition 3.11, (3.12.1) and (3.12.3) we obtain, as a corollary of Proposition 3.11, the following expression for $d_{\underline{a},\nu}$.

Proposition 3.13. *Let $\underline{a}$ be a Lie subalgebra of $\underline{g}$. Let z_i, $1 \leq i \leq n$, be an orthonormal basis of $\underline{g}$ such that for $i \leq n$, z_i is a basis of $\underline{a}$. Let $d_{\underline{a},\nu}$ be the operator on $\Lambda \underline{g} \otimes V$ given by (3.13.1). Then*

$$d_{\underline{a},\nu} = \frac{1}{2}\left(c_2 + \sum_{i=1}^{m} (\epsilon(z_i^*) \otimes 1)\theta_\nu(z_i) - \sum_{j=m+1}^{n} (\epsilon(z_j^*) \otimes 1)\theta_\nu(z_j)\right.$$

where c_2 is given by (3.12.2) and θ_ν is the tensor product of θ and ν

The significance of the operator $d_{\underline{a},\nu}$ for a family of Lie subalgebras $\underline{a}$ of $\underline{g}$ which we call Lie summands (see § 4.1) is made clear by Lemma 4.1.

4. The Laplacian in the case of a Lie summand. 1. Let $\underline{r} \subseteq \underline{g}$ be a subspace of $\underline{g}$. Define

$$\underline{r}^0 = \left\{ z \in \underline{g} \mid (z,y) = 0 \quad \text{for all } y \in \underline{r} \right\} .$$

Now let $\underline{a}$ be a Lie subalgebra of $\underline{g}$. We will say that $\underline{a}$ is a Lie summand of $\underline{g}$ if $\underline{a}^0$ is also a Lie subalgebra of $\underline{g}$.

The name Lie summand is derived from the following immediate proposition.

Proposition 4.1. Let $\underline{a}$ be a Lie subalgebra of $\underline{g}$. Then $\underline{a}$ is a Lie summand if and only if $\underline{a}^\perp$ is a Lie subalgebra of $\underline{g}$.

Proof. The proof is an immediate consequence of the obvious fact that $\underline{a}^\perp = (\underline{a}^0)^*$ and that by Remark 3.9 $\underline{a}^0$ is a Lie subalgebra if and only if $(\underline{a}^0)^*$ is a Lie subalgebra.

We recall that the representation π of § 3.1 is here the restriction of ν to $\underline{a}$. The following lemma states that in case $\underline{a}$ is a Lie summand the cobounding operator d_π on $\Lambda \underline{a}' \otimes V$ corresponds to the restriction of $d_{\underline{a},\nu}$ to $\Lambda \underline{a}^* \otimes V$ under the mapping $\eta \otimes 1$.

Lemma 4.1. Let $\underline{a}$ be a Lie summand of $\underline{g}$. Then for any $p \in \Lambda \underline{a}' \otimes V$

$$\eta \otimes 1(d_\pi p) = d_{\underline{a},\nu}(\eta \otimes 1(p))$$

Proof. By (3.12.4) and by the definition of $d_{\underline{a},\nu}$ and d_π it suffices only to show that

$$\eta \otimes 1(d_1 p) = d_{\underline{a},1}(\eta \otimes 1(p))$$

But this is an immediate consequence of Lemma 3.8, Lemma 3.10 with $\underline{r}$ replaced by $\underline{a}$, and Prop. 4.1. QED.

4.2. Now, as an operator on $\Lambda \underline{g} \otimes V$, put $c_1 = c \otimes 1$ and let

$$c_\nu = c_1 + c_2 \quad .$$

In the case of a Lie summand the problem of finding a suitable expression for the operator on $\Lambda \underline{a}^* \otimes V$ which corresponds (under $\eta \otimes 1$) to d_π is settled by Proposition 3.13 and Lemma 4.1. The corresponding problem for d_π^* is much easier. It is settled for all Lie subalgebras $\underline{a}$ by

Lemma 4.2. Let $\underline{a}$ be a Lie subalgebra of $\underline{g}$. Let $p \in \Lambda \underline{a}' \otimes V$. Then

$$\eta \otimes 1(d_\pi^* p) = c_\nu^* (\eta \otimes 1(p))$$

Proof. We first prove

$$\eta \otimes 1(d_2^* p) = c_2^*(\eta \otimes 1(p)) \qquad (4.2.1)$$

To do this first observe that $c_2^* = d_{\underline{a},2}^* + d_{\underline{a}^\perp,2}^*$ (see (3.12.2)). Next we note that $\Lambda \underline{a}^* \otimes V$ is stable under $d_{\underline{a},2}^*$ and $d_{\underline{a}^\perp,2}^*$. This is clear since both of these operators, by (3.5.1) and (3.9.3), are a sum of operators of the form $\iota(y) \otimes \nu(z)$

where $y,z \in \underline{g}$ and by (3.6.3) $\Lambda \underline{a}^* \otimes V$ is stable under every operator of this form. But now since $\Lambda \underline{a}^* \otimes V$ is stable under $d_{\underline{a},2}$ and its adjoint $d^*_{\underline{a},2}$ and since $\eta \otimes 1$ is an isometry mapping $\Lambda \underline{a}' \otimes V$ onto $\Lambda \underline{a}^* \otimes V$ it must follow from (3.12.4) that

$$\eta \otimes 1(d^*_2 p) = d^*_{\underline{a},2}(\eta \otimes 1(p))$$

To prove (4.2.1) therefore, it suffices only to show that $d^*_{\underline{a}^\perp,2}$ vanishes on $\Lambda \underline{a}^* \otimes V$. But $d^*_{\underline{a}^\perp,2}$ is a sum of operators of the form $\iota(y) \otimes \pi(z)$ where $y \in \underline{a}^\perp$. Therefore for any $x \in \underline{a}^*$ one has $(y,x) = \{y,x^*\} = 0$. Consequently by (3.6.3) $\iota(y)$ vanishes on $\Lambda \underline{a}^*$ and hence $d^*_{\underline{a}^\perp,2}$ vanishes on $\Lambda \underline{a}^* \otimes V$. This proves (4.2.1). To conclude the proof one need only show that $\eta \otimes 1(d^*_1 p) = c^*_1(\eta \otimes 1(p))$ or more simply

$$\eta\, d^* g = c^* \eta g \qquad (4.2.2)$$

for any $g \in \Lambda \underline{a}'$ since V is not involved. But now $c^* = \gamma$ by (3.9.2) and since $\underline{a}^*$ is a Lie subalgebra of $\underline{g}$ (by Remark 3.9) it follows that $\Lambda \underline{a}^*$ is stable under c^*. Therefore one need only show that for any $f \in \Lambda \underline{a}'$

$$\{\eta\, d^* g, \eta f\} = \{c^* \eta g, \eta f\}$$

But $\{\eta\, d^* g, \eta f\} = \{d^* g, f\} = \{g, df\} = \{\eta g, \eta\, df\}$. On the other hand $\{c^* \eta g, \eta f\} = \{\eta g, c\, \eta f\} = \{\eta g, \Gamma_{\underline{a}^*} c\, \eta f\}$. But $\eta(df) = \Gamma_{\underline{a}^*} c\, \eta(f)$ by Lemma 3.8. QED.

4.3. It follows immediately from Lemma 3.10 that $c_\nu = d_{\underline{g},\nu}$. Replacing $\underline{a}$ by $\underline{g}$ (obviously $\underline{g}$ is a Lie summand)

in Lemma 4.1 it then follows that c_ν is equivalent under a linear mapping to the coboundary operator of the cochain complex $C(\underline{g},V)$. Consequently $c_\nu^2 = 0$. Moreover one also knows that the relation (3.9.5) generalizes to

$$(\iota(z) \otimes 1)c_\nu + c_\nu(\iota(z) \otimes 1) = \theta_\nu(z) \qquad (4.3.1)$$

Indeed (4.3.1) is an easy consequence of (3.9.5) and the easily verified relation

$$\epsilon(y)\iota(z) + \iota(z)\epsilon(y) = (y,z)\underline{1}$$

where $y,z \in \underline{g}$ and $\underline{1}$ denotes the identity operator on $\Lambda\underline{g}$.

But now

$$(\theta_\nu(z))^* = \theta_\nu(z^*) \qquad (4.3.2)$$

This is an obvious consequence of (3.5.1) and (3.9.7)

Thus if we take the adjoint of (4.3.1) we obtain

$$(\epsilon(z^*) \otimes 1)c_\nu^* + c_\nu^*(\epsilon(z^*) \otimes 1) = \theta_\nu(z^*) \qquad (4.3.3)$$

Obviously $c_\nu^2 = 0$ implies $(c_\nu^*)^2 = 0$. It follows then from (4.3.3) after replacing z^* by z that

$$c_\nu^* \theta_\nu(z) = \theta_\nu(z)c_\nu^* \qquad (4.3.4)$$

for all $z \in \underline{g}$.

4.4. Let $R^\nu \in \text{End } V$ denote the Casimir operator corresponding to $(\underline{g})$ and the representation ν of $\underline{g}$ on V. If z_i, $1 \leq i \leq n$, is an orthonormal basis of $\underline{g}$ we note that R^ν may be written

$$R^{\nu} = \sum_{i=1}^{n} \nu(z_i^*)\nu(z_i) \qquad (4.4.1)$$

This is clear since $(z_i, z_j^*) = \delta_{ij}$.

Lemma 4.4. One has on $\Lambda \underline{g} \otimes V$ the following relation :

$$c_2 c_\nu^* + c_\nu^* c_2 = 1 \otimes R^{\nu}$$

Proof. Let z_i, $1 \leq i \leq n$, be an orthonormal basis of $\underline{g}$. Note that since $1 \otimes \nu(z_i)$ commutes with c_1^*

$$[1 \otimes \nu(z_i), c_\nu^*] = [1 \otimes \nu(z_i), c_2^*]$$

$$= \sum_j \iota(z_j^*) \otimes \nu([z_i, z_j]), \qquad (4.4.2)$$

since

$$c_2^* = \sum_j \iota(z_j^*) \otimes \nu(z_j)$$

by (3.12.1) and (3.12.2) after substituting the orthonormal basis z_j^* for the basis z_j. However since $\sum_y z_j \otimes z_j^* \in \underline{g} \otimes \underline{g}$ is invariant under the adjoint representation of $\underline{g}$ on $\underline{g} \otimes \underline{g}$ it follows from (4.4.2) that

$$[1 \otimes \nu(z_i), c_\nu^*] = \sum_j \iota([z_j^*, z_i]) \otimes \nu(z_j) \qquad (4.4.3)$$

But

$$c_2 c_\nu^* + c_\nu^* c_2 = \sum_i ((\epsilon(z_i^*) \otimes 1)c_\nu^* + c_\nu^*(\epsilon(z_i^* \otimes 1))1 \otimes \nu(z_i)$$

$$+ \sum_i (\epsilon(z_i^*) \otimes 1)[1 \otimes \nu(z_i), c_\nu^*] \qquad (4.4.4)$$

$$= \sum_i \theta_\nu(z_i^*)(1 \otimes \nu(z_1)) + \sum_j \left(\sum_i \epsilon(z_i^*) \iota([z_j^*, z_i]) \right) \otimes \nu(z_j)$$

by (4.3.3) and (4.4.3).

On the other hand

$$\sum_i \epsilon(z_i^*)\, \iota([z_j^*, z_i]) = -\theta(z_j^*) \qquad (4.4.5)$$

since both sides of (4.4.5) are derivations (see (§ 3.6)) of degree 0 of $\Lambda \underline{g}$ and both sides are easily seen to agree on $\Lambda^1 \underline{g}$. Thus by (4.4.4)

$$c_2 c_\nu^* + c_\nu^* c_2 = \sum_i (\theta_\nu(z_i^*) - \theta(z_i^*) \otimes 1)(1 \otimes \nu(z_i))$$

$$= 1 \otimes R^\nu \quad . \qquad \text{QED.}$$

We can now give an expression for the operator on $\Lambda \underline{a}^* \otimes V$ which corresponds under $\eta \otimes 1$ to the Laplacian L_π on $\Lambda \underline{a}' \otimes V$

Theorem 4.4. *Let $\underline{a}$ be a Lie summand of $\underline{g}$ (see § 4.1). Let L_π be the Laplacian on the cochain complex $C(\underline{a}, V)$ defined as in § 3.5. Let η be the mapping defined as in § 3.4. Let z_i, $1 \le i \le n$, be an orthonormal basis of $\underline{g}$ such that z_i for $i \le m$ is a basis of $\underline{a}$. Then for any $p \in \Lambda \underline{a}' \otimes V$*

$$\eta \otimes 1(L_\pi p) = \frac{1}{2}\Big(1 \otimes R^\nu + \sum_{i=1}^{m} \theta_\nu(z_i^*)\theta_\nu(z_i) - \sum_{j=1}^{m+1} \theta_\nu(z_j^*)\theta_\nu(z_j)\Big)(\eta \otimes 1(p))$$

where θ_ν is the tensor product of the adjoint representation θ and ν and R^ν is the Casimir operator corresponding to ν.

Proof. By Lemmas 4.1 and 4.2

$$\eta \otimes 1\ (L_\pi p) = (d_{\underline{a},\nu} c_\nu^* + c_\nu^* d_{\underline{a},\nu})(\eta \otimes 1(p)) \quad .$$

But now substituting the expression for $d_{\underline{a},\nu}$ given by Proposition 3.13 in $d_{\underline{a},\nu} c_\nu^* + c_\nu^* d_{\underline{a},\nu}$ and recalling that c_ν^* commutes with $\theta_\nu(z_i)$ (see (4.3.4)) the result follows from (4.3.3) and Lemma 4.4. QED.

5. The spectral resolution of the Laplacian and cohomology for a family of nilpotent Lie summands. 1. Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let ℓ (the rank of g) be its dimension. One knows that the restriction $(\underline{h})$ of $(\underline{g})$ to $\underline{h}$ is non-singular and hence one can define a map $\mu \longrightarrow x_\mu$ of $\underline{h}'$ onto $\underline{h}$ by the relation

$$(x, x_\mu) = \langle x, \mu \rangle$$

for all $x \in \underline{h}$. On the other hand, the mapping defines a non-singular bilinear form $(\underline{h}')$ on $\underline{h}'$ given by

$$(\mu, \lambda) = \langle x_\mu, \lambda \rangle \quad .$$

Now let $\Delta \subseteq h'$ be the set of roots associated with a $\underline{h}$ and let e_φ, $\varphi \in \Delta$, be a corresponding set of root vectors so that for any $\varphi \in \Delta$, $x \in \underline{h}$

$$[x, e_\varphi] = \langle x, \varphi \rangle e_\varphi \tag{5.1.1}$$

One knows that the e_φ can be chosen so that

$$\begin{aligned} (e_\varphi, e_\psi) &= 0 \quad \text{if } \psi \neq -\varphi \\ &= 1 \quad \text{if } \psi = -\varphi \end{aligned} \tag{5.1.2}$$

In such a case it is immediate that

$$[e_\varphi, e_{-\varphi}] = x_\varphi \tag{5.1.3}$$

If $\underline{h}^*$ is the real subspace of $\underline{h}'$ spanned over $\underline{R}$, by Δ, we recall that $(\underline{h}')$ is positive definite on h^*.

If $\underline{r} \subseteq \underline{g}$ is a subspace which is stable under $\theta(x)$ for all $x \in \underline{h}$ we will let $\Delta(\underline{r}) \subseteq \Delta$ be the subset defined so that

$$\underline{r} = \underline{r} \cap \underline{h} + \sum_{\varphi \in \Delta(r)} (e_\varphi)$$

Thus if $\underline{u} \subseteq \underline{g}$ is a Lie subalgebra then $\Delta(\underline{u})$ is defined if $\underline{h}$ lies in the normalizer of $\underline{u}$. In particular then $\Delta(\underline{u})$ is defined if $\underline{h} \subseteq \underline{u}$.

Denote by $\dot{+}$ the operation of addition in Δ in case the sum again lies in Δ. It is clear that $\Delta(\underline{u})$ is closed under $\dot{+}$ in case $\underline{u} \in \underline{g}$ is a Lie subalgebra.

Now let G be a simply-connected group whose Lie algebra is $\underline{g}$.

Remark 5.1. If $U \subseteq G$ is the subgroup corresponding to a Lie subalgebra $\underline{u} \subseteq \underline{g}$ and $\underline{u}$ contains a Cartan subalgebra of $\underline{g}$ then U is necessarily closed. To prove this it suffices to show that $\underline{u}$ equals its own normalizer in $\underline{g}$. But, using (5.1.1), this is immediate.

5.2. Let $\underline{b} \subseteq \underline{g}$ be a maximal solvable Lie subalgebra which will be regarded as fixed once and for all.

Let $\underline{U}$ be the collection of all Lie subalgebras $\underline{u}$ such that $\underline{b} \subseteq \underline{u}$. If $U \subseteq G$ is the subgroup corresponding to

$\underline{u} \in \underline{U}$, then it is due to Wang [8] that (see Remark 5.1) the space (left cosets, aU, $a \in G$)

$$X = G/U \tag{5.2.1}$$

is compact, has positive Euler characteristic and one obtains, up to a biholomorphic map, all such complex homogeneous spaces of G this way. Incidentally one knows also that X is algebraic (admits a holomorphic embedding into complex projective space) and that (over all $\underline{g}$) one obtains, up to a biholomorphic map, all simply connected algebraic homogeneous spaces this way.

Obviously $\underline{b} \in U$. Let Y denote the generalized flag manifold

$$Y = G/B \tag{5.2.2}$$

where $B \subseteq G$ is the subgroup corresponding to $\underline{b}$.

5.3. Let $\underline{u} \in \underline{U}$ and put

$$\underline{g}_1 \equiv \underline{u} \cap \underline{u}^* \tag{5.3.1}$$

It is clear that $\underline{g}_1$ is a Lie subalgebra of $\underline{g}$ and that $\underline{g}_1$ is closed under the $*$-operation so that (see Remark 3.9) $\underline{g}_1$ is reductive in $\underline{g}$.

Now put $\underline{m} = \underline{b}^{\circ}$. One knows that $\underline{m}$ is a maximal nilpotent Lie subalgebra of $\underline{g}$ and that $\underline{m}$ is the set of all nilpotent elements in $\underline{b}$. We note then that $\underline{b}$ is a Lie summand. This, however, is a special case of

Proposition 5.3. *Let $\underline{u} \in \underline{U}$. Then $\underline{u}$ is a Lie summand of $\underline{g}$. In fact if $\underline{n} = \underline{u}^{o}$ then $\underline{n}$ is both the maximal nilpotent ideal in $\underline{u}$ and the set of all nilpotent elements in the radical of $\underline{u}$. Furthermore if $\underline{g}_1$ is defined by (5.3.1) then*

$$\underline{g} = \underline{n}^* + \underline{g}_1 + \underline{n} \tag{5.3.2}$$

is an orthogonal direct sum and

$$\underline{u} = \underline{g}_1 + \underline{n} \tag{5.3.3}$$

Moreover $\underline{g}_1$ lies in the normalizer of both $\underline{n}$ and $\underline{n}^*$.

Proof. It is obvious from (3.9.4) that $\underline{n}$ is stable under $\underline{\theta}(z)$ for all $z \in \underline{u}$. But $\underline{b} \subseteq \underline{u}$. Hence one must have

$$\underline{n} \subseteq \underline{m} \subseteq \underline{b} \subseteq \underline{u} \tag{5.3.4}$$

Thus $\underline{n}$ is an ideal in $\underline{u}$ which proves, in particular, that $\underline{u}$ is a Lie summand. Now since $\underline{u}^* = (\underline{n}^{o})^* = n^{\perp}$ it follows, by definition, that $\underline{g}_1$ is the orthocomplement of $\underline{n}$ in $\underline{u}$. This proves (5.3.3). Furthermore $\underline{n}^* = (\underline{u}^{o})^* = \underline{u}^{\perp}$. and this proves (5.3.2).

Now let $\underline{c}$ be the center of $\underline{g}_1$ so that $\underline{s} = \underline{c} + \underline{n}$ is the radical of $\underline{u}$. But now the center of $\underline{u}$ is zero since $\underline{u}$ contains $\underline{b}$. Thus $[z,\underline{n}] = 0$, for $z \in \underline{c}$, implies $z = 0$. But since $\theta(z)$ is diagonalizable for any $z \in \underline{c}$ this implies that $\underline{n}$ is the maximal nilpotent ideal of $\underline{u}$. Furthermore since $\underline{n} \subseteq \underline{m}$ it is clear that the elements of $\underline{n}$ are nilpotent. On the other hand by simultaneously triangulizing $\theta(x)$ for all

$x \in \underline{s} = \underline{c} + \underline{n}$ it becomes obvious that $\underline{n}$ is the set of all nilpotent elements in $\underline{s}$.

Since $\underline{n}$ is an ideal of $\underline{u}$ it follows that $\underline{g}_1$ lies in the normalizer of $\underline{n}$. Applying the $*$-operation it also lies in the normalizer of $\underline{n}^*$. QED.

5.4. Now let ℓ be the rank of $\underline{g}$ and let $r = \dim \underline{m}$ so that

$$\dim \underline{g} = \ell + 2r$$

and (5.4.1)

$$\dim \underline{b} = \ell + r$$

It follows from (5.4.1) that $\dim \underline{b} \cap \underline{b}^* \geq \ell$. But since $\underline{b} \cap \underline{b}^*$ is a Lie subalgebra which is both reductive in $\underline{g}$ and solvable it follows that it is commutative. Hence $\underline{b} \cap \underline{b}^*$ is a Cartan subalgebra of $\underline{g}$. From this point we fix the Cartan subalgebra $\underline{h}$ of § 5.1 so that

$$\underline{h} = \underline{b} \cap \underline{b}^* \quad .$$

It follows then that e^*_φ is a root vector for $-\varphi$. Hence by (5.1.2) we may choose the root vectors e_φ so that in addition to (5.1.2) they form an orthonormal basis of $h^\perp$. It is immediate that this is equivalent to (5.1.2) and the condition

$$e^*_\varphi = e_{-\varphi} \tag{5.4.2}$$

for all $\varphi \in \Delta$.

Now put $\Delta_+ = \Delta(\underline{m})$ and $\Delta_- = -\Delta(\underline{m})$. One knows then that (a) $\Delta = \Delta_+ \cup \Delta_-$ is a disjoint union and . .

(b) Δ_+ (and hence Δ_-) is closed under $\dot{+}$.

Let $\Pi \subseteq \Delta_+$ be the set of simple roots corresponding to Δ_+. For any $\varphi \in \Delta$ one has

$$\varphi = \sum_{\alpha \in \Pi} n_\alpha(\varphi)\alpha \qquad (5.4.3)$$

where the $n_\alpha(\varphi)$ are non-negative or non-positive integers according as $\varphi \in \Delta_+$ or Δ_-.

Now let $\underline{u} \in \underline{U}$. Then since $\underline{h} \subseteq \underline{u}$ one knows $\underline{u}$ as soon as one knows $\Delta(\underline{u})$ or in fact $\Delta(\underline{u}) \cap \Delta_-$. The structure of these sets is well known (see [8], 7.4) and is given as follows: There are 2^{ℓ} elements in $\underline{U}$. Furthermore there is a one-one mapping, $\underline{u} \longrightarrow \Pi(\underline{u})$, of $\underline{U}$ onto the set of all subsets of Π such that

$$\Delta(\underline{u}) \cap \Delta_- = \left\{\varphi \in \Delta_- \mid n_\alpha(\varphi) = 0 \text{ for all } \alpha \in \Pi(\underline{u})\right\} \qquad (5.4.4)$$

Now let $\underline{u} \in \underline{U}$ and let $\underline{g}_1$ and $\underline{n}$ be defined as in Proposition 5.3. Since $\underline{h} \subseteq \underline{g}_1$ obviously $\Delta(\underline{g}_1)$ and $\Delta(\underline{n})$ are defined and then since $\underline{g}_1 \cap \underline{h} = \underline{h}$ and $\underline{n} \cap \underline{h} = 0$ it follows that $\underline{g}_1$ and $\underline{n}$ are determined by $\Delta(\underline{g}_1)$ and $\Delta(n)$. Clearly $\Delta(\underline{u}) = \Delta(\underline{g}_1) \cup \Delta(\underline{n})$ is a disjoint union. The following proposition is then an immediate consequence of (5.1.2), (5.4.2) and (5.4.4).

Proposition 5.4. *Let $\underline{u} \in \underline{U}$ and let $\underline{g}_1$ and $\underline{n}$ be defined as in Proposition 5.3. Then $\Delta(\underline{g}_1)$, $\Delta(\underline{n})$ and $\Delta(\underline{n}^*)$ are defined and*

$$\Delta(\underline{g}_1) = \{\varphi \in \Delta \mid n_\alpha(\varphi) = 0 \text{ for all } \alpha \in \Pi(\underline{u})\}$$
$$\Delta(\underline{n}) = \{\varphi \in \Delta_+ \mid n_\alpha(\varphi) > 0 \text{ for some } \alpha \in \Pi(\underline{u})\}$$
$$\Delta(\underline{n}^*) = -\Delta(\underline{n})$$

5.5. Let $Z \subseteq \underline{h}^* \subset \underline{h}'$ be the set of all integral linear forms on $\underline{h}$. We recall that $\mu \in Z$ if and only if $2(\mu,\varphi)/(\varphi,\varphi)$ is an integer for any $\varphi \in \Delta$.

Let $\underline{u} \in \underline{U}$ and let $\underline{g}_1$ be defined by (5.3.1). Then $\underline{h}$ is a Cartan subalgebra of $\underline{g}_1$ as well as $\underline{g}$. Let W_1 be the Weyl group of $\underline{g}_1$ regarded as operating in $\underline{h}$ and, contragrediently, also in $\underline{h}'$ (so that $\sigma(X_\mu) = X_{\sigma\mu}$ for any $\mu \in \underline{h}'$).

Let $G_1 \subseteq G$ be the subgroup (closed, see Remark 5.1) corresponding to $\underline{g}_1$. One knows that the elements of Z are the weights of all the finite dimensional representations of G_1. If ν_1 is an irreducible representation of G_1 an extremal weight of ν_1 is by definition a weight that becomes highest for some lexicographical ordering in $\underline{h}^*$. If ξ is such a weight one knows that the collection $\{\sigma\xi\}$, $\sigma \in W_1$, is the set of all extremal weights. Now for any $\xi \in Z$ let

$$\nu_1^\xi : G \longrightarrow \text{End } V_1^\xi$$

be the unique, up to equivalence, irreducible representation of G_1 having ξ as an extremal weight. Thus if $\xi_1, \xi_2 \in Z$ $\nu_1^{\xi_1}$ and $\nu_1^{\xi_2}$ are equivalent if and only if there exists $\sigma \in W_1$ such that $\sigma\xi_1 = \xi_2$.

Now let

$$\underline{m}_1 = \underline{m} \cap \underline{g}_1 \tag{5.5.1}$$

so that $\Delta(\underline{m}_1)$ is defined and

$$\Delta_+ = \Delta(\underline{m}_1) \cup \Delta(\underline{n}) \tag{5.5.2}$$

is a disjoint union where, as in § 5.3, $\underline{n} = \underline{u}^o$. Define

$$D_1 = \{\mu \in Z \mid (\mu, \varphi) \geq 0 \text{ for all } \varphi \in \Delta(\underline{m}_1) \tag{5.5.3}$$

The elements of D_1 will be called dominant (with respect to $\underline{m}_1$). One knows that D_1 is a fundamental domain for the action of W_1 on Z so that every irreducible representation of G_1 is equivalent to ν_1^{ξ} for one and only one $\xi \in D_1$. If ν_1 is an irreducible representation of G_1 the unique weight that is both dominant and extremal is called the highest weight of ν_1. Thus for any $\xi \in D_1$, ξ is the highest weight of ν_1^{ξ}. Also if V_1 is a representation space for G_1 we will call weight vectors in V_1 extremal (resp. highest) if they be in an irreducible component of V_1 and there correspond to an extremal (resp. the highest) weight.

Now if

$$\beta : G_1 \longrightarrow \text{End } C \tag{5.5.4}$$

is a representation of G_1, define $C^{\xi} \subseteq C$ as the space of all vectors in C which transform according to the irreducible representation ν_1^{ξ} of G. Thus

$$C = \sum_{\xi \in D_1} C^{\xi} \tag{5.5.5}$$

is a direct sum.

A representation of a Lie group induces a representation of its Lie algebra. We will, throughout, use the same letter, in this case β, to denote this corresponding representation of its Lie algebra. Now put

$$C_{\underline{m}_1} = \left\{ s \in C \mid \beta(z)s = 0 \text{ for all } z \in \underline{m}_1 \right\}$$

and let $C^{\xi}_{\underline{m}_1} = C_{\underline{m}_1} \cap C^{\xi}$. Then one knows that

$$C_{\underline{m}_1} = \sum_{\xi \in D_1} C^{\xi}_{\underline{m}_1}.$$

and that the set of non-zero elements in $C^{\xi}_{\underline{m}_1}$ is the set of all highest weight vectors in C^{ξ}. Thus $C^{\xi}_{\underline{m}_1}$ is the weight space for the weight ξ in the subrepresentation $\beta \mid C^{\xi}$ and

$$\dim C^{\xi}_{\underline{m}_1} = \text{multiplicity of } \nu_1^{\xi} \text{ in } \beta \qquad (5.5.6)$$

The above notation without the subscript 1 refers to the case when $\underline{g}$ is substituted for $\underline{g}_1$. Note that $D \subseteq D_1$ and that W_1 is a subgroup of W. Furthermore if we put

$$g_1 = \frac{1}{2} \sum_{\varphi \in \Delta(\underline{m}_1)} \varphi$$

then

$$g = g_1 + g_2 \qquad (5.5.7)$$

where

$$g_2 = \frac{1}{2} \sum_{\varphi \in \Delta(\underline{n})} \varphi \qquad (5.5.8)$$

For each $\varphi \in \Delta$ let $\tau_{\varphi} \in W$ be the reflection corresponding to φ so that for any $x' \in \underline{h}'$

$$\tau_\varphi x' = x' - \frac{2(x',\varphi)}{(\varphi,\varphi)}\,\varphi \qquad (5.5.9)$$

We now observe

Lemma 5.5. Let $g_2 \in h'$ be defined by (5.5.8). Then $(g_2,\varphi) = 0$ for all $\varphi \in \Delta(\underline{g}_1)$ so that $x_{g_2} \in h$ lies in the center of $\underline{g}_1$.

Proof. It follows immediately from Proposition 5.3 or Proposition 5.4 that

$$\Delta(\underline{g}_1) \dot{+} \Delta(\underline{n}) \subseteq \Delta(\underline{n})$$

Thus if $\varphi \in \Delta(\underline{g}_1)$ it follows from (5.5.8) that $\Delta(\underline{n})$ is stable under τ_φ. But then obviously $\tau_\varphi(g_2) = g_2$. Hence again by (5.5.8), $(g_2,\varphi) = 0$. QED.

Remark 5.5. Since the elements τ_φ, $\varphi \in \Delta(g_1)$, generate W_1 it follows from the proof above that $\Delta(\underline{n})$ is stable under any $\tau \in W_1$.

5.6. Let β be given by (5.5.4). Let $R^\beta \in$ End C be the Casimir operator corresponding to the restriction of $(\underline{g})$ to $\underline{g}_1$.

Let $|x'|$ denote the length of a vector $x' \in h^*$ with respect to the restriction of (h') to h^*.

We recall the following well known proposition.

Proposition 5.6. Let β be given by (5.5.4). Then, for any $\xi \in D_1$, R^β reduces to the scalar

$$|g_1 + \xi|^2 - |g_1|^2$$

on the subspace C^ξ of C.

Proof. Writing $\beta(e_\varphi)\beta(e_{-\varphi}) = \beta(x_\varphi)+\beta(e_{-\varphi})\beta(e_\varphi)$ it follows that

$$R^\beta = \sum_{i=1}^{\ell} \beta(x_i^*)\beta(x_i) + 2\beta(x_{g_1}) + 2 \sum_{\varphi \in \Delta(\underline{m}_1)} \beta(e_{-\varphi})\beta(e_\varphi)$$

where x_i, $1 \leq i \leq \ell$, is an orthonormal basis of $\underline{h}$. One knows that R^β reduces to a scalar on C^ξ so that to find the scalar it is enough to restrict R^β to $C^\xi_{\underline{m}_1}$. But since $\beta(e_{-\varphi})\beta(e_\varphi)$ vanishes on $C_{\underline{m}_1}$ the proof follows immediately. QED.

5.7. We now return to the considerations of § 4. We now assume, however, that the representation ν of $\underline{g}$ is the irreducible representation ν^λ where $\lambda \in D$, so that $V = V^\lambda$. Let $\underline{u} \in \underline{U}$ and now let $\underline{a} = \underline{n}$ where $\underline{n}$ is the nilpotent Lie summand $\underline{u}^o$. Thus $\pi = \nu^\lambda|\underline{n}$. We wish to determine the spectral resolution of the Laplacian L_π on the cochain complex $C(\underline{n},V^\lambda)$.

Now since $\underline{g}_1$ lies in the normalizer of $\underline{n}$ it follows that $\Lambda\underline{n}$ is stable under the representation $\theta|\underline{g}_1$ of $\underline{g}_1$. Let

$$\beta : \underline{g}_1 \longrightarrow \text{End } C(\underline{n},V^\lambda) \qquad (5.7.1)$$

be the representation of $\underline{g}$, on $C(\underline{n},V^\lambda)$ formed by taking the tensor product of $\nu^\lambda|\underline{g}_1$ and the representation of $\underline{g}_1$ on $\Lambda\underline{n}'$ contragredient to the representation of $\underline{g}_1$ on $\Lambda\underline{n}$ defined by restricting $\theta|\underline{g}_1$ to $\Lambda\underline{n}$. For any $z \in \underline{g}_1$ it is then obvious that, as mappings from $C(\underline{n},V)$ into $\Lambda\underline{g} \otimes V$,

$$\eta \otimes 1\ \beta(z) = \theta_\nu(z)\cdot\eta \otimes 1 \qquad (5.7.2)$$

Since β clearly arises from a representation of G_1 on $C(\underline{n},V)$ we may, as in §5.5, form the decomposition (5.5.5).

The spectral resolution of the Laplacian L_π is given in

Theorem 5.7. *Let $\lambda \in D$ and let ν^λ be an irreducible representation of $\underline{g}$ on a vector space V^λ. Let $\underline{u}$ be any Lie subalgebra of $\underline{g}$ which contains the maximal solvable Lie subalgebra $\underline{b}$ of $\underline{g}$. Let $\underline{n}$ be the maximal nilpotent ideal of $\underline{u}$ (see Proposition 5.3). Let $\pi = \nu^\lambda|\underline{n}$ and let L_π be the Laplacian on the cochain complex $C(\underline{n},V^\lambda)$ defined as in § 3.5 with $\underline{a} = \underline{n}$ and $V = V^\lambda$.*

Let $\underline{g}_1 = \underline{u} \cap \underline{u}^*$ (see § 5.3) and let β be the representation of $\underline{g}_1$ on $C(\underline{n},V^\lambda)$ defined following (5.7.1). Write, as in § 5.5,

$$C(\underline{n},V^\lambda) = \sum_{\xi \in D_1} C(\underline{n},V^\lambda)^\xi$$

where $C(\underline{n},V^\lambda)^\xi$ is the set of all vectors in $C(\underline{n},V^\lambda)$ which transform under β according to the irreducible representation ν_1^ξ of $\underline{g}_1$.

Then L_π reduces to a scalar on $C(\underline{n},V^\lambda)^\xi$ and the scalar is

$$\tfrac{1}{2}\left(|g + \lambda|^2 - |g + \xi|^2\right)$$

where

$$g = \tfrac{1}{2} \sum_{\varphi \in \Delta_+} \varphi$$

and Δ_+ is defined as in § 5.4.

Proof. Since $\underline{n}$ is a Lie summand we can apply Theorem 4.4. But by Proposition 5.3 $\underline{n}^\perp = n^* + \underline{g}_1$. Thus we can choose the

z_i of Theorem 4.4 so that for $i \leq m$, $z_i = e_\varphi$ for some $\varphi \in \Delta(\underline{n})$ and for $j > m$, z_j either lies in $\underline{n}^*$ or $\underline{g}_1$. Furthermore if $z_j \in \underline{n}^*$ then by Proposition 5.4 we may assume $z_j = e_{-\varphi}$ for some $\varphi \in \Delta(\underline{n})$.

Now apply Theorem 4.4. Then since $e^*_\varphi = e_{-\varphi}$ it follows from (5.7.2), and the definition of R^β that for any $p \in C(\underline{n}, V^\lambda)$

$$(\eta \otimes 1)L_\pi p = \frac{1}{2}\Big(1 \otimes R^{\nu_\lambda} + \sum_{\varphi \in \Delta(\underline{n})} \theta_\nu(e_{-\varphi})\,\theta_\nu(e_\varphi) - \sum_{\varphi \in \Delta(\underline{n})} \theta_\nu(e_\varphi)\theta_\nu(e_{-\varphi})\Big)\,(\eta \otimes 1(p)) - \frac{1}{2}\eta \otimes 1(R^\beta p)$$

$$= \frac{1}{2}\,(1 \otimes R^{\nu_\lambda} - 2\theta_\nu(x_{g_2}))\;\eta \otimes 1(p) - \frac{1}{2}\,\eta \otimes 1(R^\beta p)$$

where g_2 is given by (5.5.8).

But $1 \otimes R^{\nu_\lambda}$, by Proposition 5.6, reduces to the scalar $|g + \lambda|^2 - |g|^2$ on $\Lambda \underline{g} \otimes V$. Thus since $x_{g_2} \in \underline{g}_1$ we can apply (5.7.2) once more and obtain

$$L_\pi = \frac{1}{2}\,(|g + \lambda|^2 - |g|^2)\underline{1} - (\beta(x_{g_2}) + \frac{1}{2}\,R^\beta) \qquad (5.7.3)$$

where $\underline{1}$, here, denotes the identity operator on $C(\underline{n}, V^\lambda)$. But g_2 lies in the center of $\underline{g}_1$ by Lemma 5.5. Hence L_π reduces to a scalar on $C(\underline{n}, V^\lambda)^\xi$. To determine the scalar it suffices to compute L_π on a highest weight vector $p \in C(\underline{n}, V^\lambda)^\xi$. But then since p belongs to the weight ξ it follows from Proposition 5.6 that L_π reduces, on $C(\underline{n}, V^\lambda)^\xi$, to the scalar

$$\frac{1}{2}\left(|g+\lambda|^2-|g|^2\right)-\left((g_2,\xi)+(g_1,\xi)+\frac{(\xi,\xi)}{2}\right),$$

$$=\frac{1}{2}\left(|g+\lambda|^2-|g+\xi|^2\right),$$

since $g = g_1 + g_2$. QED.

Now, as one easily shows, $\beta(z)$ for any $z \in \underline{g}_1$, commutes with both d_π and d_π^*. In fact since $\beta(z)^* = \beta(z^*)$ (by 5.7.2) this is implied by Lemma 4.2, (4.3.4) and (5.7.2). Thus if we consider the orthogonal direct sum decomposition (see Remark 2.3)

$$C(\underline{n},V^\lambda) = \text{Im } d_\pi + \text{Im } d_\pi^* + \text{Ker } L_\pi \qquad (5.7.4)$$

it follows that each of the three subspaces of $C(\underline{n},V^\lambda)$ appearing on the right side of (5.7.4) is stable under $\beta(z)$ for all $z \in \underline{g}_1$ and hence induces subrepresentations of β. Since d_π maps Im d_π^*, bijectively, onto Im d_π it follows that the subrepresentations of β defined by Im d_π and Im d_π^* are equivalent.

Now since $\beta(z)$ commutes with d_π for all $z \in \underline{g}_1$ it follows that β induces a representation

$$\hat{\beta} : \underline{g}_1 \longrightarrow \text{End } H(\underline{n},V^\lambda)$$

of $\underline{g}_1$ on the cohomology space $H(\underline{n},V^\lambda)$. On the other hand it is obvious that $\hat{\beta}$ is equivalent to the subrepresentation of β defined by Ker L_π. But then since L_π is positive semi-definite we obtain, immediately, the following corollary of Theorem 5.7.

Corollary 5.7. *Let* $\xi^{\circ} \in D_1$. *Then if the multiplicity of* ν_1^{ξ} *in* β *is positive one must have*

$$|g + \lambda| \geq |g + \xi|$$

Furthermore if $|g + \lambda| > |g + \xi|$ *then the*

multiplicity of ν_1^{ξ} *in* $\hat{\beta} = 0$

and if $|g + \lambda| = |g + \xi|$ *then the*

multiplicity of ν_1^{ξ} *in* $\hat{\beta}$ = *multiplicity of* ν_1^{ξ} *in* β.

Remark 5.7. A. Another way of expressing the statement in Corollary 5.7 is as follows. If $p \in C(\underline{n}, V^{\lambda})^{\xi}$, $p \neq 0$, then p is a cocycle which is not cohomologous to zero if and only if $|g + \lambda| = |g + \xi|$. If on the other hand p is a cocycle then p is a coboundary if and only if $|g + \lambda| > |g + \xi|$.

At a later point we will make important (for us) use of the following fact (contained implicitly in Corollary 5.7).

Remark 5.7. B. Every irreducible component of $\hat{\beta}$ is inequivalent to any irreducible component of the subrepresentation of β defined by Im d_{π}.

5.8. Let $Z_+ \subseteq Z$ be the semi-group generated by Δ_+. Writing an element $\psi \in Z$ as a linear combination of simple roots it is clear that Z_+ can be characterized by

$$Z_+ = \{\psi \in Z \mid (\mu, \psi) \geq 0 \quad \text{for all} \quad \mu \in D\} \tag{5.8.1}$$

Now let $\lambda \in D$ and let Δ^{λ} denote the set of weights of the irreducible representation ν^{λ} of $\underline{g}$. One knows that if $\mu \in Z$ then a necessary condition for $\mu \in \Delta^{\lambda}$ is that

$$\lambda - \mu \in Z_+ . \tag{5.8.2}$$

The following lemma is a consequence of this fact.

Lemma 5.8. Let $\lambda_1, \lambda_2 \in D$. Let $\mu_1 \in \Delta^{\lambda_1}, \mu_2 \in \Delta^{\lambda_2}$. Then

$$|\lambda_1 + \lambda_2| \geq |\mu_1 + \mu_2| \tag{5.8.3}$$

and equality holds in (5.8.3) if and only if there exists $\sigma \in W$ such that

$$\sigma(\lambda_1 + \lambda_2) = \mu_1 + \mu_2 .$$

Proof. Let $\tau \in W$ be such that $\tau(\mu_1 + \mu_2) \in D$. For $i = 1,2$, put $\psi_i = \lambda_i - \tau\mu_i$. Since $\tau\mu_i \in \Delta^{\lambda_i}$ it follows then from (5.8.2) that $\psi_i \in Z_+$ and hence $\psi \in Z_+$ where $\psi = \psi_1 + \psi_2$. Now put $\mu = \tau\mu_1 + \tau\mu_2$ so that $\mu \in D$. But then $\lambda_1 + \lambda_2 = \mu + \psi$. Consequently, since $|\mu| = |\mu_1 + \mu_2|$, one has

$$|\lambda_1 + \lambda_2|^2 = |\mu_1 + \mu_2|^2 + |\psi|^2 + 2(\mu, \psi).$$

But by (5.8.1) $(\mu, \psi) \geq 0$. This proves the inequality (5.8.3). Furthermore if equality holds in (5.8.3) then obviously $\psi = 0$. But since $\psi = \psi_1 + \psi_2$ and $\psi_1, \psi_2 \in Z_+$ it follows that $\psi_1 = \psi_2 = 0$. That is, $\lambda_i = \tau\mu_i$, $i = 1,2$. The lemma follows in one direction by putting $\sigma = \tau^{-1}$. The other direction is obvious. QED.

Remark 5.8. Let the notation be as in Lemma 5.8. Let $\sigma \in W$. Then the proof of Lemma 5.8 also yields the statement (by putting $\tau = \sigma^{-1}$) that $\sigma(\lambda_1 + \lambda_2) = \mu_1 + \mu_2$ implies $\sigma\lambda_1 = \mu_1$ and $\sigma\lambda_2 = \mu_2$.

5.9. We recall that an element $\mu \in Z$ is called regular if $(\mu, \varphi) \neq 0$ for any $\varphi \in \Delta$. One knows that $\mu \in Z$ is regular if and only if $\sigma\mu = \mu$, $\sigma \in W$, implies σ is the identity element of W.

We recall that $g \in D$ and that g is regular. In fact both of these statements are consequences of the well known relation

$$(g,\alpha) = \frac{(\alpha,\alpha)}{2} \tag{5.9.1}$$

for any $\alpha \in \Pi$. One obtains (5.9.1) from the easily verified fact that α is the only root in Δ_+ which "changes sign" under τ_α. That is,

$$\tau_\alpha \Delta_- \cap \Delta_+ = (\alpha).$$

Consequently $\tau_\alpha g = g - \alpha$. But by (5.5.9) this is equivalent to (5.9.1).

Remark 5.9. Freudenthal has proved (see e.g. [1], 4.9)

$$|g + \lambda| > |g + \mu| \tag{5.9.2}$$

for any $\lambda \in D$ and any $\mu \in \Delta^\lambda$, $\mu \neq \lambda$.
We observe that, since g is regular and $g \in D$, (5.9.2) follows from Lemma 5.8 by putting $\lambda_1 = \mu_1 = g$.

We now wish to consider the irreducible representation ν^g of $\underline{g}$ whose highest weight is g. Weyl has given a formula for the dimension of a representation in terms of its highest weight. Weyl's formula asserts that for any $\lambda \in D$

$$\dim V^{\lambda} = \frac{\prod_{\varphi \in \Delta_+} (g + \lambda, \varphi)}{\prod_{\varphi \in \Delta_+} (g, \varphi)} \qquad (5.9.3)$$

This formula generally proves to be quite awkward for computational purposes. However in the special case when $\lambda = g$ we observe that (5.9.3) immediately yields

$$\dim V^{g} = 2^{r} \qquad (5.9.4)$$

where r (= dim $\underline{m}$) is the number of roots in Δ_+.

We wish to determine the weights of V^{g} and their multiplicities. For any subset $\Phi \subseteq \Delta_+$ let $\langle \Phi \rangle \in Z$ be defined by

$$\langle \Phi \rangle = \sum'_{\varphi \in \langle \Phi \rangle} \varphi \quad .$$

Let the elements of Δ_+ be ordered so that $\Delta_+ = \{\varphi_i\}$, $i = 1, 2, \ldots, r$. Now observe that if $\Phi \subseteq \Delta_+$

$$g - \langle \Phi \rangle = \tfrac{1}{2} (\pm \varphi_1 \pm \varphi_2 \pm \cdots \pm \varphi_r) \qquad (5.9.5)$$

for some choice of the signs and that furthermore as Φ runs through all 2^r subsets of Δ_+ then the right hand side of (5.9.5) runs through all 2^r choices of signs. It is suggestive from (5.9.4) and definition of g that V^{g} somehow behaves like a spin representation. The analogy is further strengthened by

Lemma 5.9. Let $f \in Z$. Then $f \in \Delta^g$ if and only if there exists $\Phi \subseteq \Delta_+$ such that

$$f = g - \langle \Phi \rangle \tag{5.9.6}$$

Furthermore the multiplicity of f as a weight of ν^g is equal totthe number of subsets $\Phi \subseteq \Delta_+$ satisfying (5.9.6).

Proof. Let $s \subseteq \text{End } g$ be the Lie algebra of all operators on g which are skew-symmetric with respect to (g). Thus s is isomorphic to the Lie algebra of $SO(n,C)$. Since $\theta(z)$, for $z \in g$, is determined by its restriction to g we may regard θ as a monomorphism mapping g into s. Furthermore we may find a Cartan subalgebra d of s such that θ maps h into d.

Now let

$$\theta' : d' \longrightarrow h'$$

be the mapping whose transpose is the restriction of θ to h. It is obvious that if ρ is a representation of s then θ' maps the weights of ρ into the weights of $\rho \circ \theta$.

Now one knows that the non-zero weights of the given representation of s on g are of the form $\pm \lambda_i$, $i = 1,2,\ldots,\left[\frac{n}{2}\right]$, where $\lambda_1,\ldots,\lambda_{\left[\frac{n}{2}\right]}$ are linearly independent in d'. Furthermore it is clear that we can choose the ordering and signs of the λ_i so that

$$\theta'(\lambda_i) = \begin{cases} \varphi_i , & i = 1,2,\ldots,r \\ 0 , & i = r+1,\ldots,\left[\frac{n}{2}\right] \end{cases} \tag{5.9.6}$$

Now let

$$\upsilon : \underline{s} \longrightarrow \text{End } V^{\upsilon}$$

be the spin representation of $\underline{s}$. One knows that $\dim V^{\upsilon} = 2^{\left[\frac{n}{2}\right]}$ and that the weights of υ are all elements in $\underline{d}'$ of the form

$$\tfrac{1}{2}\left(\pm\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_{\left[\frac{n}{2}\right]}\right)$$

and that each weight occurs with multiplicity one. Writing $n = \ell + 2r$ it follows then from (5.9.5) and (5.9.6) that the weights of $\upsilon \circ \theta$ are all elements $f \in \underline{h}'$ of the form $g - \left\langle \Phi \right\rangle$ where $\Phi \subseteq \Delta_+$ and that the multiplicity of f is equal to $2^{\left[\frac{\ell}{2}\right]}$ times the number of subsets $\Phi \subseteq \Delta_+$ such that

$$f = g - \left\langle \Phi \right\rangle$$

In particular we note that g is a weight of $\upsilon \circ \theta$ and that its multiplicity is at least $2^{\left[\frac{\ell}{2}\right]}$. On the other hand we now observe that every weight vector corresponding to g is necessarily a highest weight vector. To prove this it suffices to note that if $\varphi \in \Delta_+$ then $g + \varphi$ is not a weight of $\upsilon \circ \theta$. Indeed if it were we would have $g + \varphi = g - \left\langle \Phi \right\rangle$ or $\varphi + \left\langle \Phi \right\rangle = 0$ for a subset $\Phi \subseteq \Delta_+$. But this is impossible since $Z_+ \cap -Z_+ = 0$. This proves that that the multiplicity of v^g in $\upsilon \circ \theta$ is at least $2^{\left[\frac{\ell}{2}\right]}$. But from the identity

$$2^{\left[\frac{n}{2}\right]} = 2^{\left[\frac{\ell}{2}\right]} 2^r$$

it follows from (5.9.4) that v^g occurs exactly $2^{\left[\frac{\ell}{2}\right]}$ times

in $\upsilon \circ \theta$ and that no other irreducible representation of $\underline{g}$ occurs in $\upsilon \circ \theta$. The lemma then follows from the statement above concerning the weights of $\upsilon \circ \theta$. QED.

5.10. Let $\sigma \in W$. Define the subset $\Phi_\sigma \subseteq \Delta_+$ by putting

$$\Phi_\sigma = \sigma\Delta_- \cap \Delta_+$$

It then follows at once that

$$\sigma g = g - \langle \Phi_\sigma \rangle \qquad (5.10.1)$$

Since $\sigma g \in \Delta^g$ and, being extremal, since it occurs with multiplicity one as a weight of v^g it follows from (5.10.1) and Lemma 5.9 that for any subset $\Phi \subseteq \Delta_+$

$$\langle \Phi \rangle = \langle \Phi_\sigma \rangle \text{ implies } \Phi = \Phi_\sigma \qquad (5.10.2)$$

Now one knows (see e.g. [1], 4.9) that the mapping $\sigma \longrightarrow \sigma\Delta_-$ is a bijection of W onto the family of all subsets Δ_0 of Δ satisfying the two conditions(1) Δ_0 is closed under $\dot{+}$ and (2) $\Delta = \Delta_0 \cup -\Delta_0$ is a disjoint union. It follows then that there exists a unique element $\mathscr{w} \in W$ such that

$$\Phi_{\mathscr{w}} = \Delta_+ \qquad (5.10.3)$$

Furthermore one deduces

Proposition 5.10. The mapping $\sigma \longrightarrow \Phi_\sigma$ of W is a bijection of W onto the family of all subsets Φ of Δ_+ which

satisfy the condition that Φ and its complement Φ^c in Δ_+ are both closed under $\dot{+}$.

Proof. Since g is a regular element of $\underline{h}'$ (see (5.9.1)) it follows immediately from (5.10.1) that the mapping $\sigma \longrightarrow \Phi_\sigma$ is an injection.

Now by definition it is obvious that Φ_σ is closed under $\dot{+}$. But since the complement of Φ_σ in Δ_+ is equal to $\sigma\Delta_+ \cap \Delta_+ = \sigma(\mathcal{H}\Delta_-) \cap \Delta_+ = \Phi_{\sigma\mathcal{H}}$ it follows that it too, is closed under $\dot{+}$.

Conversely assume that $\Phi \subseteq \Delta_+$ and its complement Φ^c in Δ_+ are both closed under $\dot{+}$. Put

$$\Delta_o = \Phi \cup -(\Phi^c) \quad .$$

Obviously $\Delta = \Delta_o \cup -\Delta_o$ is a disjoint union. On the other hand it is straightforward to verify that Δ_o is closed under $\dot{+}$. Hence, as noted above, $\Delta_o = \sigma\Delta_-$ for some unique $\sigma \in W$. But then obviously $\Phi = \Phi_\sigma$. QED.

For later use we record the fact, noted in the above proof, that, if $\mathcal{H} \in W$ is given by (5.10.3),

$$\Delta_+ = \Phi_\sigma \cup \Phi_{\sigma\mathcal{H}} \tag{5.10.4}$$

is a disjoint union for any $\sigma \in W$.

5.11. For any subset $\Phi \subseteq \Delta_+ \ (= \Delta(\underline{m}))$ denote by $e_\Phi \in \bigwedge \underline{m}$ the element given by

$$e_\Phi = e_{\varphi_{i_1}} \wedge \ldots \wedge e_{\varphi_{i_k}}$$

where $\Phi = \{\varphi_{i_1}, \ldots, \varphi_{i_k}\}$. It is obvious that the elements e_Φ, $\Phi \subseteq \Delta_+$, form a basis of $\Lambda \underline{m}$ and if we put

$$e_{-\Phi} = e^*_\Phi \tag{5.11.1}$$

then the elements $e_{-\Phi}$, $\Phi \subseteq \Delta_+$ form a basis of $\Lambda \underline{m}^*$.

Now let

$$\zeta : \underline{h} \longrightarrow \text{End } \Lambda \underline{m}^* \otimes V^\lambda$$

be the representation of $\underline{h}$ on $\Lambda \underline{m}^* \otimes V^\lambda$ obtained by restricting $\theta_\nu | \underline{h}$ to $\Lambda \underline{m}^* \otimes V^\lambda$. Let Δ^ζ be the set of weights of ζ. It is obvious then that if $\xi \in Z$ then $\xi \in \Delta^\zeta$ if and only if ξ can be written as

$$\xi = -\langle \Phi \rangle + \mu \tag{5.11.2}$$

where $\Phi \subseteq \Delta_+$ and $\mu \in \Delta^\lambda$. But then as an immediate corollary to Lemma 5.9 we obtain

Lemma 5.11. *Let $\xi \in Z$. Then $\xi \in \Delta^\zeta$ if and only if $g + \xi$ can be written*

$$g + \xi = f + \mu$$

where $f \in \Delta^g$ and $\mu \in \Delta^\lambda$.

5.12. For any $\sigma \in W$ put

$$\xi_\sigma = \sigma(g + \lambda) - g \tag{5.12.1}$$

Also let $s_{\sigma\lambda} \in V^\lambda$ be the extremal weight vector (unique up to a scalar multiple) corresponding to the weight $\sigma\lambda$ of V^λ.

The following lemma is the main lemma needed together with Theorem 5.7 to yield the cohomology group $H(\underline{n}, V^{\lambda})$.

Lemma 5.12. *For any* $\xi \in \Delta^{\zeta}$ *one has*

$$|g + \lambda| \geq |g + \xi|$$

Let $\sigma \in W$ *and let* ξ_σ *be defined by* (5.12.1). *Then the mapping* $\sigma \longrightarrow \xi_\sigma$ *is a bijection of* W *onto the set of all weights* ξ *of* ζ *such that*

$$|g + \lambda| = |g + \xi|.$$

Furthermore, as a weight of ζ, ξ_σ *occurs with multiplicity one and the weight vector corresponding to* ξ_σ *is the element*

$$e_{-\Phi_\sigma} \otimes s_{\sigma\lambda}$$

of $\Lambda \underline{m}^* \otimes V^{\lambda}$.

Proof. It follows immediately from Lemmas 5.8 and 5.11 (putting $\lambda_1 = g$, $\mu_1 = f$, $\lambda_2 = \lambda$, $\mu_2 = \mu$) that $\xi_\sigma \in \Delta^{\zeta}$, that $|g + \lambda| \geq |g + \xi|$ for $\xi \in \Delta^{\zeta}$ and that equality holds if and only if $\xi = \xi_\sigma$ for some $\sigma \in W$. Also $\xi_\sigma = \xi_\tau$ implies $\sigma = \tau$ since $g + \lambda$ is obviously regular.

Since $e_{-\Phi_\sigma} \otimes s_{\sigma\lambda}$ is obviously a weight vector for ξ_σ to prove the lemma it suffices only to show that the multiplicity of ξ_σ is one. But since we can find a basis of $\Lambda \underline{m}^* \otimes V^{\lambda}$ consisting of weight vectors of the form $e_{-\Phi} \otimes s_\mu$ where $\mu \in \Delta^{\lambda}$ and $s_\mu \in V^{\lambda}$ is a corresponding weight vector it suffices only to show that

$$\xi_\sigma = - \langle \phi \rangle + \mu \qquad (5.12.2)$$

implies $\bar{\Phi} = \bar{\Phi}_\sigma$ and $\mu = \sigma\lambda$. But now if (5.12.2) is satisfied then adding g one has

$$\sigma(g + \lambda) = f + \mu$$

where $f = g - \langle \bar{\Phi} \rangle$ so that $f \in \Delta^g$. But then by Remark 5.8 $f = \sigma g$ and $\mu = \sigma\lambda$. However $f = \sigma g$ implies $\langle \bar{\Phi} \rangle = \langle \bar{\Phi}_\sigma \rangle$. But then $\bar{\Phi} = \bar{\Phi}_\sigma$ by (5.10.2). QED.

5.13. Let $\underline{u} \in \underline{U}$ and let g_1 and $\underline{n}$ be defined as in § 5.3. We isolate a subset W^1 of W by setting

$$W^1 = \{\sigma \in W \mid \bar{\Phi}_\sigma \subseteq \Delta(\underline{n}) \qquad (5.13.1)$$

Recalling (5.5.2) and (5.5.3) we observe that the elements of W^1 can be characterized as follows:

Remark 5.13. Let $\sigma \in W$. Then the following three conditions are equivalent, (1) $\sigma \in W^1$, (2) $\sigma^{-1}(\Delta(\underline{m}_1)) \subseteq \Delta_+$ and (3) $\sigma(D) \subseteq D_1$.

The following proposition states that W^1 defines a "cross-section" with respect to the canonical mapping of W onto the right coset space $W_1 \backslash W$.

Proposition 5.13. Every element $\tau \in W$ can be uniquely written

$$\tau = \tau_1 \sigma$$

where $\tau_1 \in W_1$ and $\sigma \in W^1$.

Proof. Let $\sigma_1, \sigma_2 \in W^1$. Let $\tau_1 = \sigma_1 \sigma_2^{-1}$ and assume $\tau_1 \in W_1$. Then by Remark 5.5 $\Delta(\underline{n})$ is stable under τ_1. But this clearly implies

$$\Phi_{\sigma_2^{-1}} \subseteq \Phi_{\sigma_2^{-1}} .$$

On the other hand taking inverse $\sigma_2 \sigma_1^{-1}$ also lies in W_1. Thus $\Phi_{\sigma_2^{-1}} = \Phi_{\sigma_1^{-1}}$ which, by Proposition 5.10 implies $\sigma_1 = \sigma_2$. Thus no two distinct elements of W^1 lie in the same right coset of W_1.

Now let $\tau \in W$ be arbitrary. Let $\Phi_1 = \tau(\Delta_-) \cap \Delta(\underline{m}_1)$ and let Φ_2 be the complement of Φ_1 in $\Delta(\underline{m}_1)$. Then $\Phi_2 = \tau(\Delta_+) \cap \Delta(\underline{m}_1)$ so that both Φ_1 and Φ_2 are closed under $\dot{+}$. Now apply Proposition 5.10 to the case where $[\underline{g}_1, \underline{g}_1]$ the maximal semi-simple ideal of $\underline{g}_1$ is substituted for $\underline{g}$. It follows then that there exists $\tau_1 \in W_1$ such that (since $\Delta(\underline{n}^*)$ is stable under τ_1)

$$\Phi_{\tau_1} = \Phi_1$$

Now put $\sigma = \tau_1^{-1}\tau$. It is then straightforward to verify $\sigma(\Delta_-) \cap \Delta(\underline{m}_1)$ is empty so that $\sigma \in W^1$. QED.

It is implicit in the proof above that if $\tau = \tau_1 \sigma$ is the decomposition given by Proposition 5.13 then

$$\Phi_\tau = \Phi_{\tau_1} \cup \tau_1(\Phi_\sigma) \qquad (5.13.2)$$

is a disjoint union; the components on the right being also the respective intersections of Φ_τ with $\Delta(\underline{m}_1)$ and $\Delta(\underline{n})$.

Now for any $\sigma \in W$ put

$$n(\sigma) = \text{number of roots in } \Phi_\sigma \tag{5.13.3}$$

Since, obviously,

$$\Phi_{\sigma^{-1}} = -\sigma^{-1}(\Phi_\sigma) \tag{5.13.4}$$

note that

$$n(\sigma) = n(\sigma^{-1}) \tag{5.13.5}$$

Furthermore if $\tau \in W$ and $\tau = \tau_1\sigma$ is the decomposition given by Proposition 5.13 then it follows from (5.13.2) that

$$n(\tau) = n(\tau_1) + n(\sigma) \tag{5.13.6}$$

Remark 5.13. Let $\tau \in W$. We note as a consequence of (5.13.6) that the unique element $\sigma \in W^1$ in the right coset $W_1\tau$ can be characterized by the statement that $n(\sigma) \leq n(\tau')$ for all $\tau' \in W\tau$ and that equality holds if and only if $\tau' = \sigma$. Using (5.13.5) it follows that a similar statement involving the set $\{\sigma^{-1}\}$, $\sigma \in W^1$, can be said for the left cosets of W_1.

5.14. Now for any non-negative integer j put

$$W(j) = \{\sigma \in W \mid n(\sigma) = j\}$$

and let

$$W^1(j) = W(j) \cap W^1 \quad .$$

Also let $\{e'_{-\Phi}\}$, $\Phi \subseteq \Delta(\underline{n})$, be the basis of $\Lambda \underline{n}'$ dual to the basis $\{e_{-\Phi}\}$, $\Phi \subseteq \Delta(\underline{n})$, of $\Lambda \underline{n}$ so that by (5.11.1) and (3.2.1)

$$\eta(e'_{-\Phi}) = e_{-\Phi} \tag{5.14.1}$$

We can now state

Theorem 5.14. *Let $\underline{u}$ be any Lie subalgebra of $\underline{g}$ which contains the maximal solvable Lie subalgebra $\underline{b}$ of $\underline{g}$. Let $\underline{n}$ be the maximal nilpotent ideal of $\underline{u}$ (see Proposition 5.3) and let $\underline{g}_1 = \underline{u} \cap \underline{u}^*$ so that $\underline{g}_1$ is a reductive (in $\underline{g}$) Lie subalgebra and $\underline{u} = \underline{g}_1 + \underline{n}$ is a semi-direct sum (as Lie algebras).*

Let $\lambda \in D$ and let ν be the irreducible representation of $\underline{g}$ on a vector space V^λ whose highest weight is λ.

Let $H(\underline{n},V^\lambda)$ be the cohomology group formed with respect to the representation $\pi = \nu^\lambda|_{\underline{n}}$ of $\underline{n}$ on V^λ and let $\hat{\beta}$ be the representation of $\underline{g}_1$ on $H(\underline{n},V^\lambda)$ defined as in § 5.7.

Now for any $\xi \in D_1$ let $H(\underline{n},V^\lambda)^\xi$ be the space of all classes in $H(\underline{n},V^\lambda)$ which transform under $\hat{\beta}$ according to the irreducible representation ν_1^ξ of $\underline{g}_1$ whose highest weight is ξ.

Now for any $\sigma \in W$ let ξ_σ be defined by

$$\xi_\sigma = \sigma(g + \lambda) - g.$$

Then if $\sigma \in W^1$ one has $\xi_\sigma \in D_1$ and for any $\xi \in D_1$ one has $H(\underline{n},V^\lambda)^\xi \neq 0$ if and only if $\xi = \xi_\sigma$ for some $\sigma \in W^1$. Furthermore $H(\underline{n},V^\lambda)^{\xi_\sigma}$ is irreducible for all $\sigma \in W^1$ so that $\sigma \longrightarrow H(\underline{n},V^\lambda)^{\xi_\sigma}$ is a bijection of W^1 onto the set of all irreducible (under $\hat{\beta}$) components of $H(\underline{n},V^\lambda)$. Moreover degree-wise, for any non-negative integer j

$$H^j(\underline{n},V^\lambda) = \sum_{\sigma \in W^1(j)} H(\underline{n},V^\lambda)^{\xi_\sigma}$$

(direct sum) so that for any $\sigma \in W^1$, the elements of $H(\underline{n},V^\lambda)^{\xi_\sigma}$ are homogeneous of degree $n(\sigma)$. Finally if $s_{\sigma\lambda} \in V^\lambda$ is the weight vector for the extremal weight $\sigma\lambda$ of v^λ then the highest weight in $H(\underline{n},V^\lambda)^{\xi_\sigma}$ is the cohomology class having

$$e'_{-\Phi_\sigma} \otimes s_{\sigma\lambda}$$

as a representative (harmonic) cocycle.

Proof. Now by Corollary 5.7 $H(\underline{n},V^\lambda)^{\xi} \neq 0$ if and only if ξ is a highest weight of an irreducible component of β and

$$|g + \lambda| = |g + \xi| \qquad (5.14.1)$$

Moreover in such a case the multiplicity of v_1^ξ in β is the same as its multiplicity in $\hat{\beta}$. But now the representation $\beta|\underline{h}$ of $\underline{h}$ is obviously equivalent to the subrepresentation of $\mathcal{S}$ (see § 5.11) of $\underline{h}$ defined by the subspace $\Lambda \underline{n}^* \otimes V$ of $\Lambda \underline{m}^* \otimes V$. But then by Lemma 5.12 the only weights of β which satisfy (5.14.1) are the weights ξ_σ for $\sigma \in W^1$ and they occur with multiplicity one. Therefore to prove the theorem up to the statement "Moreover ...", it suffices only to show that the ξ_σ occur as highest weights in the decomposition of β. But to prove this it is enough to show, for any $\varphi \in \Delta(\underline{m}_1)$, $\sigma \in W^1$, that $\xi_\sigma + \varphi$ is not weight of β.

Put $\xi = \xi_\sigma + \varphi$. Then

$$|g + \xi|^2 = |\sigma(g+\lambda)+\varphi|^2 = |g+\lambda|^2 + 2(\sigma(g+\lambda),\varphi)+|\varphi|^2$$

But now by Remark 5.13 (3), $\sigma(g + \lambda) \in D_1$ so that $(\sigma(g + \lambda),\varphi) \geq 0$. But then $|g + \xi| > |g + \lambda|$. By Lemma 5.12 this implies ξ is not a weight of γ and a fortiori ξ is not a weight of β.

Now by Lemma 5.12 $e'_{-\Phi_\sigma} \otimes s_{\sigma\lambda}$ is the unique (up to scalar multiple) weight vector for the weight ξ_σ of β. But from above it must be the highest weight vector of an irreducible component of β. Hence by Theorem 5.7 $e'_{-\Phi_\sigma} \otimes s_{\sigma\lambda}$ is a harmonic cocycle (element of Ker L_π). But then, clearly, its cohomology class is the highest weight vector in $H(\underline{n},V)^{\xi_\sigma}$. Now this class is obviously homogeneous of $n(\sigma)$. Since $H(\underline{n},V^\lambda)^{\xi_\sigma}$ is generated by its highest weight vector under the action of $\underline{g}_1$ it follows therefore that $H(\underline{n},V^\lambda)^{\xi_\sigma} \leq H^{n(\sigma)}(\underline{n},V^\lambda)$. This completes the proof. QED.

In our applications we are interested in the action of $\underline{g}_1$ on $H(\underline{n},V^\lambda)$, as given by Theorem 5.14, rather than in $H(\underline{n},V^\lambda)$ itself. Nevertheless as a corollary to Theorem 5.14 one obtains

Corollary 5.14. *Let $\underline{n}$, V^λ and the representation π of $\underline{n}$ on V^λ be as in Theorem 5.14. Then*

$$\dim H^j(\underline{n},V^\lambda) = \frac{\sum_{\sigma \in W^1(j)} \prod_{\varphi \in \Delta(\underline{m}_1)} (\sigma(g+\lambda),\varphi)}{\prod_{\varphi \in \Delta(\underline{m}_1)} (g,\varphi)}$$

Proof. Let $(\underline{g})_1$ be any non-singular, invariant bilinear form on $\underline{g}$ and let $(\underline{h}')_1$ be the bilinear form on $\underline{h}'$ induced by $(\underline{g})_1$. Now observe that in Weyl's formula (5.9.3) one obtains the same result using $(\underline{h}')_1$ instead of $(\underline{h}')$. (This is clear since any root corresponds to a simple component of $\underline{g}$). Furthermore one need only assume that $\underline{g}$ is reductive instead of semi-simple. But then to determine $\dim H(\underline{n},V^\lambda)^{\xi_\sigma}$ one may apply Weyl's formula to the representation $\nu_1^{\xi_\sigma}$ of $\underline{g}_1$ using the restriction $(\underline{g}_1)$ of $(\underline{g})$ to $\underline{g}_1$. But for any $\varphi \in \Delta(\underline{m}_1)$,

$$(\sigma(g+\lambda) - g + g_1,\varphi) = (\sigma(g+\lambda),\varphi)$$

and

$$(g_1,\varphi) = (g,\varphi)$$

since by Lemma 5.5 $(g-g_1,\varphi) = (g_2,\varphi) = 0$. The corollary then follows from Theorem 5.14. QED.

5.15. Let $w(j)$ (resp. $w^1(j)$) be the number of elements σ in $W(j)$ (resp. $W^1(j)$).

In general the dimension of $H^j(\underline{n},V^\lambda)$ varies with λ. In fact by Corollary 5.14, if $\underline{m}_1 \neq 0$, by choosing λ properly it can be made to be arbitrarily large. However if $\underline{m}_1 = 0$, that is, if $\underline{m}$ is substituted for $\underline{n}$ it was first proved by

Bott ([2]) that dim $H^j(\underline{m},V^{\lambda})$ is constant over all $\lambda \in D$. In fact he observed that

$$\dim H^j(\underline{m},V^{\lambda}) = w(j) \qquad (5.15.1)$$

This result of Bott is an immediate consequence of

Corollary 5.15. Let the notation be as in Theorem 5.14 except that $\underline{m}$ is substituted for $\underline{n}$ so that W is substituted for W^1. Then, for any $\sigma \in W$, $H(\underline{n},V^{\lambda})^{\xi_\sigma}$ is one dimensional and in fact

$$H(\underline{n},V^{\lambda})^{\xi_\sigma} = ((e'_{-\Phi_\sigma} \otimes s_{\sigma\lambda}))$$

where $(e'_{-\Phi_\sigma} \otimes s_{\sigma\lambda})$ is the cohomology class defined by the cocycle $e'_{-\Phi_\sigma} \otimes s_{\sigma\lambda}$.

Proof. In the special case of Theorem 5.14 considered here $\underline{h}$ plays the role of $\underline{g}_1$. But since $H(\underline{n},V^{\lambda})^{\xi_\sigma}$ is irreducible under $\underline{h}$ and since $\underline{h}$ is commutative it follows that dim $H(\underline{n},V^{\lambda})^{\xi_\sigma}$ is one dimensional. QED.

Remark. 5.15. Observe that a statement generalizing the result (5.15.1) to the case of $\underline{n}$ involves multiplicity of representations rather than dimension. Such a statement is the following: The number of irreducible components in $H^j(\underline{n},V^{\lambda})$ under the action of $\underline{g}_1$ is equal to $w^1(j)$ (and consequently is independent of λ).

6. Applications I. The generalized Borel-Weil Theorem.1. Let $\underline{u} \subset \underline{U}$ (see § 5.2) and let $\underline{n}$ $(= \underline{u}^0)$ be the maximal nilpotent ideal in $\underline{u}$. Also, as in § 5.3 let $\underline{g}_1 = \underline{u} \cap \underline{u}^*$. Now let U, N and G_1 be the subgroups of G corresponding, respectively to $\underline{u}$, $\underline{n}$ and $\underline{g}_1$. The subgroups U and G, we recall, are closed by Remark 5.1 . But N is closed also since $\theta(N)$ is unipotent (see Proposition 5.3). Thus since the center of G_1 operates reductively on $\underline{g}$ it is clear that $G_1 \cap N$ reduces to the identity and hence

$$U = G_1 N \tag{6.1.1}$$

is a semi-direct product.

Since $\underline{n}$ lies in the commutator of $\underline{u}$ (because $\underline{h} \subset \underline{u}$) it is clear that N maps onto a unipotent linear group under any representation of U. But since N is normal in U it is obvious then that any irreducible representation of U is trivial on N and hence is equivalent to ν_1^{ξ} , for some $\xi \in D_1$, on G_1. Conversely given $\xi \in D_1$ or, more generally, given $\xi \in Z$ (see § 5.5) the representation ν_1^{ξ} of G_1 on V_1^{ξ} extends to an irreducible representation

$$\nu_1^{\xi} : U \longrightarrow \text{End } V_1^{\xi}$$

of U on V_1^{ξ} by making it trivial on N. Hereafter we will regard ν_1^{ξ} as so extended. Thus, up to equivalence all irreducible representations of U are of the form ν_1^{ξ} for $\xi \in D_1$.

Now, as in § 5.2, let $X = G/U$. Then X may be regarded as the base space of a holomorphic fiber bundle with G as

total space and U as fiber. Given $\xi \in Z$ one obtains an associated holomorphic vector bundle E^{ξ} with fiber V_1^{ξ} as the set of equivalence classes in $G \times V_1^{\xi}$ with respect to the equivalence relation

$$(au,s) \equiv (a, v_1^{\xi}(u)s)$$

for any $a \in G$, $u \in U$ and $s \in V_1^{\xi}$.

Let $a,b \in G$. If $x = bU \in X$ let $a \cdot x \in X$ denote the coset abU. Similarly if $v \in E^{\xi}$ is the equivalence class containing (b,s) where $s \in V_1^{\xi}$, let $a \cdot v \in E^{\xi}$ denote the equivalence class containing (ab,s). It is clear then that if $X_o \subseteq X$ is an open set in X and ψ is a local holomorphic section of E^{ξ} defined on $a^{-1} \cdot X_o$ then $a(\psi)$, given by

$$a(\psi)(x) = a \cdot \psi(a^{-1} \cdot x) ,$$

where $x \in X_o$ is a local holomorphic section of E^{ξ} defined on X_o. But now the mapping $\psi \longrightarrow a(\psi)$ defines an operator $\rho^{\xi}(a)$ on $H(X, \underline{SE}^{\xi})$ where $\underline{SE}^{\xi}$ is the sheaf of local holomorphic sections of E^{ξ} and $H(X, \underline{SE}^{\xi})$ is the cohomology group over X with coefficients in $\underline{SE}^{\xi}$.

Now from general considerations concerning such cohomology groups one knows that $H(X, \underline{SE}^{\xi})$ is finite dimensional. But for any $j = 0,1,\ldots,$ and $a \in G$, it is obvious that $H^j(X, \underline{SE}^{\xi})$ is stable under $\rho^{\xi}(a)$. We will let

$$\rho^{j,\xi} : G \longrightarrow \text{End } H^j(X, \underline{SE}^{\xi})$$

be the representation of G (and also $\underline{g}$) defined by restricting $\rho^{\xi}(a)$ to $H^j(X,\underline{SE}^{\xi})$ for all $a \in G$.

It is clear, using Weyl's dimension formula, that a knowledge as to how $\rho^{j,\xi}$ decomposes into irreducible representations yields in particular the dimension of $H^j(X,\underline{SE}^{\xi})$. We concern ourselves then with the question of decomposing $\rho^{j,\xi}$.

6.2. Let $\xi, \lambda \in Z$. Let ρ_1 be the representation of $\underline{u}$ on Hom (V^{λ}, V_1^{ξ}), the space of linear mappings from V^{λ} into V_1^{ξ}, defined by putting

$$\rho_1(y)(A) = \nu_1^{\xi}(y)A - A\nu^{\lambda}(y) \qquad (6.2.1)$$

for all $y \in \underline{u}$ and all $A \in$ Hom (V^{λ}, V_1^{ξ}). Then with respect to this representation one can form the relative cohomology group $H(\underline{u},\underline{g}_1,$ Hom $(V^{\lambda}, V_1^{\xi}))$. Concerning this cohomology group and the decomposition of $\rho^{j,\xi}$ Bott (see [2], 1.6) has proved

Proposition 6.2. *Let $\xi \in Z$. For $j = 0,1,\ldots$ let $\rho^{j,\xi}$ be the representation of G on $H^j(X,\underline{SE}^{\xi})$ defined in § 6.1. Then for any $\lambda \in Z$ one has*

$$\text{mult. of } \nu^{\lambda} \text{ in } \rho^{j,\xi} = \dim H^j(\underline{u},\underline{g}_1, \text{ Hom } (V^{\lambda}, V_1^{\xi}))$$

In the next section we will put Proposition 6.2 in a some somewhat simpler form (Proposition 6.3) expressing it as a reciprocity law.

6.3. Now one knows that for any $\xi \in Z$ the representation $\nu_1^{-\xi}$ is equivalent to the representation contragredient to ν_1^{ξ}. Without loss of generality therefore we will, from now on, assume that, for any $\xi \in Z$, $V_1^{-\xi}$ is in fact the dual space of V_1^{ξ} and $\nu_1^{-\xi}$ is the representation contragredient ν_1^{ξ}.

For any $\xi \in Z$ the unique extremal weight of ν_1^{ξ} lying in $-D_1$ will be called the lowest weight of ν_1^{ξ} (corresponding weight vectors are called lowest weight vectors). Thus for any $\xi \in D_1$ one has that $-\xi$ is the lowest weight of $\nu_1^{-\xi}$. (In this section it will be convenient to use $-D_1$ (as we may) instead of D_1 to index the irreducible representations of G_1).

Substituting G for G_1 the conventions made above will hold also when ν^{λ} is substituted for ν_1^{ξ}.

Let $\lambda \in D$ and $\xi \in D_1$. Now in the usual manner we may identify Hom $(V^{-\lambda}, V_1^{-\xi})$ with $V^{\lambda} \otimes V^{-\xi}$. It is clear then that ρ_1, for the values $-\lambda$, $-\xi$, is equal to the tensor product of $\nu^{\lambda}|\underline{u}$ and $\nu_1^{-\xi}$. On the other hand let

$$\nu_1 : \underline{u} \longrightarrow \text{End } V_1$$

be any representation such that $\nu_1|\underline{g}_1$ is completely reducible and let $H(\underline{n},V_1)$ be defined with respect to $\nu_1|\underline{n}$. Then where the representation

$$\hat{\beta}_1 : \underline{g}_1 \longrightarrow \text{End } H(\underline{n},V_1)$$

is defined in a manner similar to the definition of $\hat{\beta}$ in § 5.7 and $H^j(\underline{n},V_1)^0$ is the set of all elements in $H^j(\underline{n},V_1)$

transforming under $\hat{\beta}$, according to the zero representation of $\underline{g}_1$ it is a simple and well known fact (see e.g. [2], Corollary 2, p. 223 or [5], p. 603) that

$$\dim H^j(\underline{n},V_1)^0 = \dim H^j(\underline{u},\underline{g}_1,V_1) \tag{6.3.1}$$

for $j = 0,1,\ldots,$.

Now putting $V_1 = \mathrm{Hom}\,(V^{-\lambda},V^{-\xi})$ and ν_1 equal to the tensor product of $\nu^{\lambda}|\underline{u}$ and $\nu_1^{-\xi}$ and recalling that $\nu_1^{-\xi}|\underline{n}$ is trivial it follows that

$$H^j(\underline{n},V_1) = (H^j(\underline{n},V^{\lambda})) \otimes V_1^{-\xi} \tag{6.3.2}$$

Now let

$$\hat{\beta}^{j,\lambda} : \underline{g}_1 \longrightarrow \mathrm{End}\, H^j(\underline{n},V^{\lambda})$$

be the representation of $\underline{g}_1$ on $H^j(\underline{n},V^{\lambda})$ defined by restricting $\hat{\beta}$ (see § 5.7) to $H^j(\underline{n},V^{\lambda})$. It then follows from (6.3.1) and (6.3.2) that

$$\dim H^j(\underline{u},\underline{g}_1,V_1) = \text{mult. of } \nu_1^{\xi} \text{ in } \hat{\beta}^{j,\lambda} \tag{6.3.3}$$

Substituting $-\lambda$ for λ and $-\xi$ for ξ, Proposition 6.2 becomes the following reciprocity law.

Proposition 6.3. *Let j be a non-negative integer. Let $\xi \in D_1$ and let $\rho^{j,-\xi}$ be the representation of $\underline{g}$ on $H^j(X,\underline{S}E^{-\xi})$ defined as in § 6.1. Let $\lambda \in D$ and let $\hat{\beta}^{j,\lambda}$ be the representation of $\underline{g}_1$ on $H^j(\underline{n},V^{\lambda})$ defined above.* Then

$$\text{mult. of } \nu^{-\lambda} \text{ in } \rho^{j,-\xi} = \text{mult. of } \nu_1^{\xi} \text{ in } \hat{\beta}^{j,\lambda}$$

Remark 6.3. The proof of Proposition (Bott) 6.2 may be simplified considerably. In fact after making a few simple observations the proof of Proposition 6.2 or rather more directly Proposition 6.3, follows almost immediately from a theorem of Dolbeault. We will sketch the arguments.

Let $K \subsetneq G$ be the subgroup of G corresponding to $\underline{k}$. Let $C^\infty(K)$ be the space of all infinitely differentiable complex valued functions on K. Now let ν_L and ν_R be the representations of K on $C^\infty(K)$ defined by

$$(\nu_L(a)f)(b) = f(a^{-1}b)$$

and

$$(\nu_R(a)f)(b) = f(ba)$$

where $f \in C^\infty(K)$ and $a, G \in K$.

Now the representation ν_R induces (by differentiation) a representation of $\underline{k}$ on $C^\infty(K)$ and by complexification a representation

$$\nu_R : \underline{g} \longrightarrow \text{End } C^\infty(K) \qquad (6.3.4)$$

of $\underline{g}$ on $C^\infty(K)$.

Now let

$$p : G \longrightarrow X$$

be the canonical mapping. One knows (since $\underline{g} = \underline{k} + \underline{u}$) that p maps K onto X so that p induces a diffeomorphism of K/K_1 on X where $K_1 = U \cap K$. Note that by definition of $\underline{g}_1$ (see § 5.3) one has that $\underline{g}_1$ is the complexification of $\underline{k}_1$ where $\underline{k}_1$ is the Lie algebra of K_1. It follows therefore that if $\xi \in D_1$ and

$$\nu_1 : \underline{g}_1 \longrightarrow \text{End } (C^\infty(K) \otimes V_1^{-\xi})$$

is the representation defined by taking the tensor product of $\nu_R|\underline{g}_1$ and $\nu_1^{-\xi}$ and $(C^\infty(K) \otimes V_1^{-\xi})^0$ is the set of all elements in $C^\infty(K) \otimes V_1^{-\xi}$ transforming, under ν_1, according to the zero representation of $\underline{g}_1$ then $(C^\infty(K) \otimes V_1^{-\xi})^0$ is canonically isomorphic to the space of all C^∞ cross sections of $E^{-\xi}$. This fact leads immediately to the Frobenius reciprocity law.

But now one has the following. Let $a \in K$ and let q_a be the mapping of $\underline{g}$ onto the complex tangent space to K at a induced by ν_R (6.3.4) and let p_a be the mapping, induced by p, of the complex tangent space to K at a onto the complex tangent space to X at p(a). One then observes that the composition $p_a \circ q_a$ maps $\underline{n}$ bijectively onto the set of all anti-holomorphic tangent vectors at p(a); that is, onto the space of all complex tangent vectors at p(a) which are orthogonal to the space of all holomorphic 1-covectors at p(a). It follows then that if

$$\pi_R : \underline{n} \longrightarrow \text{End } (C^\infty(K) \otimes V_1^{-\xi})$$

is the representation defined by taking the tensor product of $\nu_R|n$ and the trivial representation and if

$$\beta_R : \underline{g}_1 \longrightarrow \text{End } C(\underline{n}, C^\infty(K) \otimes V_1^{-\xi})$$

is the representation of $\underline{g}_1$ on the cochain complex $C^{-\xi} = \dot{C}(\underline{n}, C^\infty(K) \otimes V_1^{-\xi})$ (formed with respect to π_R) defined in the same way as β of § 5.7 except that ν_1 replaces $\nu^\lambda|\underline{g}_1$ then, more generally for any j, $(C^{j,-\xi})^0$ is canonically

isomorphic to the space $C^{0,j}(X,E^{-\xi})$ of all C^∞ differential forms of type $(0,j)$ on X with values in $E^{-\xi}$. Here $(C^{j,-\xi})^0$ is the space of all homogeneous elements of degree j in $C^{-\xi}$ which transform under β_R according to the zero representation of $\underline{g}_1$. (We say more generally since if $j=0$ this statement is identical with the one made above concerning $(C^\infty(K) \otimes V_1^{-\xi})^0$). Moreover if

$$d_R : (C^{j,-\xi})^0 \longrightarrow (C^{j+1,-\xi})^0$$

is the mapping induced by the coboundary operator on $C^{-\xi}$ then under the isomorphism $(C^{i,-\xi})^0 \longrightarrow C^{0,i}(X,E^{-\xi})$, $i = j, j+1$, one also observes (and this is the key observation) that d_R corresponds to the usual coboundary operator d'' on $C^{0,j}(X,E^{-\xi})$. It follows then from the reductive properties of the action of $\underline{g}_1$ that one obtains an isomorphism

$$(H^j(\underline{n},C^\infty(K) \otimes V_1^{-\xi}))^0 \equiv H^{0,j}(X,E^{-\xi}) \quad (6.3.5)$$

where the superscript 0 is defined with respect to $\hat{\beta}_R$ and $\hat{\beta}_R$ is defined in a similar manner to $\hat{\beta}$ of § 5.7.

On the other hand by Dolbeault's theorem one has the isomorphism

$$H^{0,j}(X,E^{-\xi}) \longrightarrow H^j(X,\underline{S}E^{-\xi}) \quad (6.3.6)$$

so that (6.3.5) and (6.3.6) yield the isomorphism

$$(H^j(\underline{n},C^\infty(K) \otimes V_1^{-\xi}))^0 \longrightarrow H^j(X,\underline{S}E^{-\xi}) \quad (6.3.7)$$

But now the representation ν_L of K on $C^\infty(K)$ extends to a representation $\nu_L \otimes 1$ of K on $C^\infty(K) \otimes V_1^{-\xi}$. Since $\nu_L \otimes 1$

obviously commutes with ν_1 and π_R induces a representation

$$\rho_L^{j,-\xi} : K \longrightarrow \text{End } (H^j(\underline{n}, C^\infty(K) \otimes V_1^{-\xi}))^0$$

One then observes that under the isomorphism (6.3.6) $\rho_L^{j,-\xi}$ corresponds to the representation $\rho^{j,-\xi} | K$ of K on $H^j(X, \underline{S}E^{-\xi})$ (see § 6.1).

Now one proceeds in a manner similar to the one used in the proof of the Frobenius reciprocity law. Using the Peter-Weyl decomposition of $C^\infty(K)$ one easily establishes an isomorphism

$$H^j(\underline{n}, C^\infty(K) \otimes V_1^{-\xi}))^0 \longrightarrow \sum_{\lambda \in D} V^{-\lambda} \otimes (H^j(\underline{n}, V^\lambda) \otimes V_1^{-\xi})^0 \tag{6.3.8}$$

where if $\rho^{-\lambda}$ is the representation of K on the summand $V^{-\lambda} \otimes (H^j(\underline{n}, V^\lambda) \otimes V_1^{-\xi})^0$ formed by taking the tensor product of $\nu^{-\lambda} | K$ on $V^{-\lambda}$ and the trivial representation of K on $(H^j(\underline{n}, V^\lambda) \otimes V_1^{-\xi})^0$ and ρ is the representation of K on the right hand member of (6.3.8) formed by taking the direct sum of the $\rho^{-\lambda}$ then ρ corresponds to $\rho_L^{j,-\xi}$.

But then Proposition 6.3 follows from the obvious fact, observed before, that

$$\dim (H^j(\underline{n}, V^\lambda) \otimes V_1^{-\xi})^0 = \text{mult. of } \nu_1^{\xi} \text{ in } \hat{\beta}^{j,\lambda}.$$

6.4. Let $\xi \in D_1$. One knows that $H^0(X, \underline{S}E^{-\xi})$ is just the space of all holomorphic cross-sections of $E^{-\xi}$.

Now assume that $\underline{u} = \underline{b}$, so that $X = Y$ and $D_1 = Z$. In this case $E^{-\xi}$ is a line bundle over X. It follows from a well known

theorem of Kodaira on positive line bundles that if $\xi \in D$ then

$$H^j(Y, \underline{S}E^{-\xi}) = 0 \quad \text{for all } j > 0 \ .$$

But then applying Hirzebruch's formulation of the Riemann-Roch theorem to the case at hand one obtains that

$$\dim H^0(Y, \underline{S}E^{-\xi}) = \dim V^{-\xi} \tag{6.4.1}$$

(note that on the right hand side the subscript 1 is absent). If $\rho^{0,-\xi}$ is the representation of G on $H^0(Y, \underline{S}E^{-\xi})$ defined as in § 6.1. it is then suggestive from (6.4.1)

$$\rho^{0,-\xi} \text{ is equivalent to } \nu^{-\xi}$$

It is the assertion of the theorem of Borel-Weil that this is in fact the case.

Now return to the case where $\underline{u} \in \underline{U}$ is arbitrary. (Here again one can still show, without the use of cohomology or sheaf theory, that if $\xi \in D$ then the representation $\rho^{0,-\xi}$ of G on the space $H^0(X, \underline{S}E^{-\xi})$ of holomorphic sections of $E^{-\xi}$ is equivalent to $\nu^{-\xi}$).

We now consider the general situation where $\xi \in D_1$ is arbitrary (so that the Kodaira theory is not applicable) and the nature of $\rho^{j,-\xi}$ is sought for arbitrary j.

For any $\lambda \in D$ and $\sigma \in W$ let $\xi(\lambda, \sigma) \in Z$ be defined by putting

$$\xi(\lambda, \sigma) = \sigma(g + \lambda) - g$$

In § 5.12, since λ was regarded as fixed, we denoted this element, more simply, by ξ_σ.

We now isolate a special subset D_1^0 of D_1 (and thereby, by our indexing, isolate a special family of representations of g_1). Let D_1^0 be defined by putting

$$D_1^0 = \{\xi \in D \mid g + \xi \text{ is regular (see § 5.9)}\}.$$

We first observe

Lemma 6.4. *Let W^1 be defined by (5.13.1). Then the mapping*

$$D \times W^1 \longrightarrow Z$$

given by

$$(\lambda, \sigma) \longrightarrow \xi(\lambda, \sigma),$$

where $\lambda \in D$ and $\sigma \in W^1$, maps $D \times W^1$ bijectively onto D_1^0.

Proof. Let $\xi \in D_1$. Since $g \in D \subseteq D_1$ it is obvious that $g + \xi \in D_1$. But now if $\xi \in D_1^0$ then there exists a unique $\sigma \in W$ such that $\sigma^{-1}(g + \xi) \in D$. Furthermore by Remark 5.13 (3) it is clear that $\sigma \in W^1$. Moreover since $\sigma^{-1}(g + \xi) \in D$ is regular it follows from (5.9.1) that λ also lies in D where λ is defined by

$$\lambda = \sigma^{-1}(g + \xi) - g$$

But then obviously $\xi(\lambda, \sigma) = \xi$. Moreover the uniqueness of σ obviously shows that $\xi(\lambda', \sigma') = \xi$ implies $\lambda = \lambda'$ and $\sigma = \sigma'$ if $\lambda' \in D$.

It suffices only to prove $\xi(\lambda, \sigma) \in D_1^0$ for all $\lambda \in D$ and $\sigma \in W^1$. But by Theorem 5.14 $\xi(\lambda, \sigma) \in D_1$ (one can easily give a simpler and more direct proof of this fact). On the other

hand if $\xi = \xi(\lambda, \sigma)$ then $g + \xi = \sigma(g + \lambda)$ and since $g + \lambda$ is obviously regular it follows also that $g + \xi$ is regular so that $\xi \in D_1^0$. QED.

Remark 6.4. Lemma 6.4 should perhaps be viewed in the following light. If $\xi \in D_1^0$ then writing $\xi = \xi(\lambda, \sigma)$ we observe that ξ, and hence also the representation ν_1^ξ of $\underline{g}_1$, picks out in this way a unique $\lambda \in D$ and hence a special representation ν^λ of $\underline{g}$ and also a unique $\sigma \in W^1$ and hence, in particular, a special integer $n(\sigma)$.

Note also that $D \subseteq D_1^0 \subseteq D_1$ and that if $\xi \in D$ then upon writing $\xi = \xi(\lambda, \sigma)$ one has $\lambda = \xi$ and σ is the identity element of W.

After applying the Riemann-Roch theorem in the general case considered above the following generalization of the Borel-Weil theorem was conjectured by Borel and Hirzebruch. It was then later proved by Bott (Theorem IV', [2]).

Theorem 6.4. *Let $\xi \in D_1$. Then if $\xi \notin D_1^0$ one has*

$$H^j(X, \underline{S}E^{-\xi}) = 0 \text{ for all } j = 0, 1, \ldots .$$

If $\xi \in D_1^0$ then upon writing (uniquely, see Lemma 6.4) $\xi = \xi(\lambda, \sigma)$ where $\lambda \in D$ and $\sigma \in W^1$ one has

$$H^j(X, \underline{S}E^{-\xi}) = 0 \text{ for all } j \neq n(\sigma)$$

and for $j = n(\sigma)$ one has

$$\dim H^{n(\sigma)}(X, \underline{S}E^{-\xi}) = \dim V^{-\lambda}$$

where in fact if $\rho^{n(\sigma), -\xi}$ is defined as in § 6.1 then

$\rho^{n(\sigma),-\xi}$ *is equivalent to the irreducible representation* $\nu^{-\lambda}$ *of* G.

Proof. We have only to apply Proposition 6.3, Theorem 5.14 and Lemma 6.4. That is, if $\xi \notin D_1^o$ then by Lemma 6.4 and Theorem 5.14 the mult. of $\nu_1^{-\xi}$ in $\hat{\beta}^{j,-\lambda'}$ equals zero for all $\lambda' \in D$ and all j. It follows then from Proposition 6.3 that $\nu^{-\lambda'}$ has zero multiplicity in $\rho^{j,-\xi}$ for all j and $\lambda' \in D$. Hence $\rho^{j,-\xi}$ is the zero representation for all j. This proves the first statement.

Similarly if $\xi = \xi(\lambda,\sigma) \in D_1^o$ then Lemma 6.4 and Theorem 5.14 assert that the multiplicity of $\nu_1^{-\xi}$ in $\hat{\beta}^{j,-\lambda'}$ is zero for all $\lambda' \in D$ and all j unless both $j = n(\sigma)$ and $\lambda' = \lambda$ in which case the multiplicity is one. The theorem then follows from Proposition 6.3. QED.

7. Application II. Weyl's character formula and its extension to non-connected groups. 1. In this section let U be any (not necessarily connected) complex Lie group.

Let $\underline{n}$ be the Lie algebra of a normal connected Lie subgroup of U. Let $m = \dim \underline{n}$ and let

$$\beta_o^1(a) \in \text{End } \Lambda^1 \underline{n}' \qquad (7.1.1)$$

be the inverse transpose to the automorphism of $\underline{n}$ induced from conjugation by a. Furthermore for $j = 0,1,\ldots,\dim \underline{n}$ let

$$\beta_o^j : U \longrightarrow \text{End } \Lambda^j \underline{n}'$$

be the representation of U on $\Lambda^j \underline{n}'$ formed by taking the j^{th} exterior product of the representation defined by (7.1.1).

Now for any $a \in U$ put

$$\chi_o^{(j)}(a) = \text{trace } \beta_o^j(a)$$

and let

$$\chi_o(a) = \sum_{j=0}^{m} (-1)^j \chi_o^{(j)}(a) \quad .$$

One, of course, knows that

$$\chi_o(a) = \det (1 - \beta_o^1(a)) \quad . \qquad (7.1.2)$$

Now for any subset $U' \subseteq \mathbb{U}$ let $R(U') \subseteq U'$ be defined by

$$R(U') = \left\{ a \in U' \mid \chi_o(a) \neq 0 \right\} .$$

Note that, by (7.1.2), R(U) is the set of all $a \in U$ such that $\beta_o^1(a)$ has no non-zero fixed vectors.

Remark 7.1. Although we make no use of the fact it can be easily shown that if R(U) is not empty then $\underline{n}$ is necessarily a nilpotent Lie algebra.

7.2. Now let

$$\nu : U \longrightarrow \text{End } V$$

be a representation of U on V and, for any $a \in U$, let

$$\chi^\nu(a) = \text{trace } \nu(a) \quad .$$

Our intention now is to give a formula for the character χ^ν involving cohomology groups defined by $\underline{n}$.

Let

$$\beta^j : U \longrightarrow \text{End } \Lambda^j \underline{n}' \otimes V$$

be the tensor product of the representations β_o^j and ν. Thus if χ_j is the character of β_j one obviously has

$$\chi^\nu \chi_o^{(j)} = \chi^{(j)} \tag{7.2.1}$$

Let $\pi = \nu|\underline{n}$. Then we recall that $\Lambda \underline{n}' \otimes V$ is the underlying space of the cochain complex $C(\underline{n},V)$ defined by π. Furthermore if d_π is the corresponding coboundary operator then it follows easily that for any $a \in U$

$$d_\pi \beta^j(a) = \beta^{j+1}(a) d_\pi \tag{7.2.2}$$

on $C^j(\underline{n},V)$.

Since (7.2.2) holds also for j-1, β^j induces a representation

$$\hat{\beta}^j : U \longrightarrow \text{End } H^j(\underline{n},V)$$

of U on the cohomology group $H^j(\underline{n},V)$.

Now let $\hat{\chi}^{(j)}$ be the character of $\hat{\beta}^j$ and put

$$\hat{\chi} = \sum (-1)^j \hat{\chi}^{(j)} .$$

Similarly put

$$\chi = \sum (-1)^j \chi^{(j)} .$$

It is then a simple and well known fact (the Euler-Poincare principle) using (7.2.2) that for any $a \in U$

$$\chi(a) = \hat{\chi}(a) . \tag{7.2.3}$$

Let $\hat{\chi}_o$ equal $\hat{\chi}$ for the case when the identity representation is substituted for ν. It follows then from (7.2.3) that also

$$\chi_o(a) = \hat{\chi}_o(a) \quad . , \tag{7.2.4}$$

But now taking the alternating sum with the expressions in (7.2.1) as summands one obtains

$$\chi^\nu \cdot \chi_o = \chi \quad . \tag{7.2.5}$$

We have proved, using (7.2.3), (7.2.4) and (7.2.5)

Proposition 7.2. *Let ν be a representation of U on a vector space V and χ^ν be its character.*

Let $\underline{n}$ be the Lie algebra of a normal Lie subgroup of U and let $\hat{\chi}$ (resp. $\hat{\chi}_o$) be the alternating sum of the characters of the representations $\hat{\beta}^j$ (resp. $\hat{\beta}^j_o$) of U on $H^j(\underline{n},V)$ (resp. $H^j(\underline{n})$).

Let R(U) be the set of all $a \in U$ which, (see Remark 7.1) under the representation of U on $\underline{n}$ induced by conjugacy, correspond to operators on $\underline{n}$ without non-zero fixed vectors.

Then if $a \in R(U)$ one has $\chi_o(a) \neq 0$ and

$$\chi^\nu(a) = \frac{\hat{\chi}(a)}{\hat{\chi}_o(a)} \quad .$$

If $a \notin R(U)$ one has $\hat{\chi}_o(a) = \hat{\chi}(a) = 0$.

7.3. Let $\underline{u} \in U$ (see § 5.2). We apply Proposition 7.2. to the case where U is the subgroup of G corresponding to $\underline{u}$ and $\underline{n}$ (= $\underline{u}^o$) is the maximal nilpotent ideal of $\underline{u}$. Also let

$\nu = \nu^{\lambda} \mid U$ where $\lambda \in D$ so that $V = V^{\lambda}$.

Now if $\xi \in D_1$ let χ_1^{ξ} be the character of the representation ν_1^{ξ} of G_1. Then if, as in §7.2, $\hat{\chi}^{(j)}$ is the character of the representation $\hat{\beta}^{j}$ of U on $H(\underline{n}, V^{\lambda})$ it follows from Theorem 5.14 that for any $a \in G_1 \subseteq U$.

$$\hat{\chi}^{(j)}(a) = \sum_{\sigma \in W^1(j)} \chi_1^{\xi_\sigma}(a) \qquad (7.3.1)$$

where, we recall, $\xi_\sigma = \sigma(g + \lambda) - g$ and $W^1(j)$ is given by (§ 5.14)

For any $\sigma \in W$ let $sg\sigma$, as usual denote the determinant of σ . If $n(\sigma)$ is defined by (5.13.3) it is then well known that

$$sg\sigma = (-1)^{n(\sigma)} \qquad (7.3.2)$$

(In fact since there are obviously $n(\sigma)$ root "walls" separating, for example, g and σg (7.3.2) follows from the fact that $sg\tau_\varphi = -1$ for any $\varphi \in \Delta$). But now Proposition 7.2, (7.3.1) and (7.3.2) yields

Proposition 7.3. Let $\lambda \in D$ and let χ^{λ} be the character of the irreducible representation ν^{λ} of G. Let G_1 and $\underline{n}$ be defined as in § 5.3 and let $R(G_1)$ be the set of all $a \in G_1$ such that $\Theta(a)z = z$ for $z \in \underline{n}$ implies $z = 0$. Here Θ denotes the adjoint representation of G on $\underline{g}$. Then for any $a \in R(G_1)$

$$\chi^{\lambda}(a) = \frac{\sum_{\sigma \in W^1} \operatorname{sg} \sigma \, \chi_1(a)^{\sigma(g+\lambda)-g}}{\sum_{\sigma \in W^1} \operatorname{sg} \sigma \, \chi_1(a)^{\sigma(g)-g}} \qquad (7.3.3)$$

where for any $\xi \in D_1$, χ_1^{ξ} is the character of the irreducible representation ν_1^{ξ} of G_1 and W^1 is given by (5.13.1).

7.4. Now consider the special case of Proposition 7.3 where $\underline{u} = \underline{b}$ so that $\underline{n} = \underline{m}$ and $G_1 = H$ where $H \subseteq G$ is the (Cartan) subgroup corresponding to $\underline{h}$. In this case $D_1 = Z$ and $W^1 = W$. Furthermore if $a \in H$ then writing $a = \exp x$ for $x \in \underline{h}$ one has, for any $\xi \in Z$

$$\chi_1^{\xi} = e^{\langle \xi, x \rangle}$$

Moreover R(H) is the set of all elements in H that are regular in G.

Multiplying numerator and denominator of (7.3.3) by $e^{\langle g, x \rangle}$ one obtains, as an immediate corollary to the proposition above,

Proposition 7.4. (Weyl's character formula) Let χ^{λ} be the character of the representation ν^{λ}. Let $a \in H$ be regular in G. Then writing $a = \exp x$ one has

$$\chi^{\lambda}(a) = \frac{\sum_{\sigma \in W} \operatorname{sg} \sigma \; e^{\langle \sigma(g+\lambda), x \rangle}}{\sum_{\sigma \in W} \operatorname{sg} \sigma \; e^{\langle \sigma g, x \rangle}}$$

Remark 7.4. The proof of Weyl's character formula given above, depended on the knowledge of how H operates on $H(\underline{m}, V^{\lambda})$. In determining the latter (Corollary 5.15), however, we have used (5.9.4), that is the fact that $\dim V^{g} = 2^{r}$. But our proof of (5.9.4) required the special case $\lambda = g$ of Weyl's dimension formula and the latter one derives from Weyl's character formula. To avoid circular reasoning we would like to point out that one can derive the fact that $\dim V^{g} = 2^{r}$ by a consideration of the Clifford algebra over $\underline{g}$ with respect to the bilinear form $(\underline{g})$. Such a proof, although longer, does not use Weyl's dimension formula.

Remark 7.4. Let $x \in \underline{h}$ and put $a = \exp x$. Note then that the identity $\chi^{\circ}(a) = \hat{\chi}^{\circ}(a)$, (see (7.2.4)), is just the familiar relation

$$\prod_{\varphi \in \Delta_{+}} (1 - e^{-\langle \varphi, x \rangle}) = e^{-\langle g, x \rangle} \cdot \sum_{\sigma \in W} \operatorname{sg} \sigma \; e^{\langle \sigma g, x \rangle}.$$

7.5. Let $\underline{g}^+$ be any reductive complex Lie algebra. Without loss of generality, however, we may assume that $\underline{g}$ is the maximal semi-simple ideal in $\underline{g}^+$. Now let G^+ be any complex Lie group (not necessarily connected) whose Lie algebra is $\underline{g}^+$.

Here we will let Θ denote the adjoint representation of G^+ on $\underline{g}^+$. If $a \in G^+$ it is clear of course that $\underline{c}$, the center of $\underline{g}^+$, and $\underline{g}$ are both stable under $\Theta(a)$. However, we note that both $\Theta(a)|\underline{c}$ and $\Theta)a_|\underline{g}$ may be, respectively, outer automorphisms of $\underline{c}$ and $\underline{g}$.

Now let

$$H^+ = \{a \in G^+ \mid \underline{m} \text{ and } \underline{h} \text{ are both stable under } \Theta(a)\} \quad (7.5.1)$$

In case G^+ is connected it is clear that H^+ is a Cartan subgroup of G^+. However, if G^+ is not connected then, for one thing, the group H^+ may not be commutative (and in fact H^+ may be quite complicated especially if the identity component, G_e^+, of G^+ has a non-trivial center and $\underline{g}^+$ has a large number of isomorphic simple components.). Nevertheless as far as conjugacy and representation theory are concerned, as we now observe, H^+ appears to be the natural substitute for H.

Now let C denote the Cartan group of $\underline{g}$ operating, like W, in $\underline{h}$ and contragrediently in $\underline{h}'$. One knows then (since W is transitive on the Weyl chambers) that

$$C = C_o W$$

is a semi-direct product where W, the Weyl group, is a normal subgroup of C and

$$C_o = \{\tau \in C \mid \tau : \Pi \longrightarrow \Pi\}.$$

Now let $a \longrightarrow \tau(a)$ be the homomorphism of H^+ into C_o defined by the condition that for any $a \in H^+$, $\alpha \in \Pi$

$$(\Theta(a)e_\alpha) = (e_{\tau(a)\alpha}) \tag{7.5.2}$$

We then denote by $C_o^+ \subseteq C_o$ the image of H^+ under this homomorphism.

Now let

$$\nu_1 : H^+ \longrightarrow \text{End } V_1$$

be an irreducible representation of H^+ on V_1. Since ν_1 induces a representation of $\underline{h}$ (which clearly also arises from a representation of H) we may consider the set Δ^{ν_1} of weights of ν_1 (we ignore the center $\underline{c}$ of $\underline{g}$) and note that $\Delta^{\nu_1} \subseteq Z$. It is then immediate from (7.5.1) that if $\lambda \in \Delta^{\nu_1}$ then Δ^{ν_1} is given by

$$\Delta^{\nu_1} = \{\tau\lambda \mid \tau \in C_o^+\} \tag{7.5.3}$$

We will now say that ν_1 is a dominant representation of H^+ if

$$\Delta^{\nu_1} \subseteq D .$$

Since D is stable under C_o observe that by (7.5.3) ν_1 is dominant if $\lambda \in D$ for at least one $\lambda \in \Delta^{\nu_1}$.

Now let Λ be an index set for the equivalence classes of all dominant irreducible representations of H^+.

Now just as the elements of D index both the classes of dominant representations of H and all representations of G we

now observe that Λ is an index set for the classes of all irreducible representations of G^+. That is, to each $\delta \in \Lambda$ there exists a unique (up to equivalence) irreducible representation

$$\nu^{\delta} : G^+ \longrightarrow \text{End } V^{\delta}$$

such that if

$$V_1^{\delta} = \{ s \in V^{\delta} \mid \nu^{\delta}(e_{\varphi})s = 0 \quad \text{for all } \varphi \in \Delta_+ \}$$

and

$$\nu_1^{\delta} : H^+ \longrightarrow \text{End } V_1^{\delta}$$

is the representation defined by restricting $\nu^{\delta} \mid H^+$ to V_1^{δ} (obviously a stable subspace) then ν_1^{δ} is a dominant irreducible representation of H^+ belonging to the equivalence class corresponding to δ. Furthermore every irreducible representation ν of G^+ is equivalence to ν^{δ} for some, necessarily unique, $\delta \in \Lambda$. The proof of the statements above proceed in the same way as in the classical situation as soon as one observes that $G^+ = H^+ G_e$ where G_e is the subgroup of G_e^+ corresponding to $\underline{g}$.

Now let $a \in H^+$ and let $\sigma \in W$. Since W is normal in C we can let $\sigma' \in W$ be defined by the relation

$$\tau(a)\sigma = \sigma'\tau(a).$$

Recalling that by definition $\Phi_{\sigma} = \sigma(\Delta_-) \cap \Delta_+$ we then observe that for some scalar $\chi_1^{\sigma}(a)$,

$$\theta(a)\ e^{-\underline{\Phi}_\sigma} = \chi_1^\sigma(a)\ e^{-\underline{\Phi}_{\sigma'}} \tag{7.5.4}$$

Similarly if $V_\sigma^\delta \subseteq V^\delta$ is defined in the same way as V_1^δ except that $\sigma\Delta_+$ replaces Δ_+ we observe that

$$\nu^\delta(a) : V_\sigma^\delta \longrightarrow V_{\sigma'}^\delta, \tag{7.5.5}$$

It follows therefore that if H_σ^+ is the subgroup of H' defined by

$$H_\sigma^+ = \{a \in H^+ \mid \tau(a) \text{ commutes with } \sigma\}.$$

then V_σ^δ is stable under $\nu^\delta \mid H_\sigma^+$ and hence defines a representation ν_σ^δ of H_σ^+. Let χ_σ^δ be the character of ν_σ^δ.

Remark 7.5. If χ_1^δ is the character of ν_1^δ note that χ_1^δ determines χ_σ^δ for any $\sigma \in W$. In fact if $b(\sigma) \in G_e$ is any element which induces, by conjugation, the transformation σ on $\underline{h}$ observe that

$$\chi_\sigma^\delta(a) = \chi_1^\delta(b(\sigma)ab(\sigma)^{-1})$$

for any $a \in H_\sigma^+$. The element $b(\sigma)$ is needed since σ itself does not in general operate on H_σ^+.

Finally put

$$\chi_1^{\sigma,\delta}(a) = \chi_1^\sigma(a)\ \chi_\sigma^\delta(a) \tag{7.5.6}$$

for any $a \in H_\sigma^+$.

An element $a \in G^+$ is called regular if the rank of $\theta(a) - 1$ is minimum in the connected component of G^+ containing a. In case $\underline{g}^+ = \underline{g}$ this definition is the same as that given by Gantmacher, [6], p.112,119. Since $\theta(a)|\underline{c} = \theta(b)|\underline{c}$

for a,b $\in$ G^+ lying in the same connected component we may apply the results of [6] to the case at hand, In particular it follows then from Theorems 12, 23 and 29 in [6] that every regular element is conjugate to an element in H^+ and that a $\in$ H^+ is regular if and only if the kernel of $\theta(a) - 1$ lies in $\underline{h}^+$, the Lie algebra of H^+. But the latter clearly implies that a $\in$ H^+ is regular if and only if $\theta(a)$ has no fixed vectors in m. Thus if we apply the considerations of §7.2 to the case where U is the normalizer in G^+ of the subgroup of G_e corresponding to $\underline{m}$ and $\underline{n} = \underline{m}$ it follows that $R(H^+)$ is the set of all elements in H^+ that are regular in G^+.

Applying Proposition 7.2 where $v = v^\delta | U$ we obtain the following generalization of Weyl's character formula.

Theorem 7.5. Let $\underline{g}^+$ be any reductive Lie algebra and let G^+ be any Lie group (not necessarily connected) whose Lie algebra is $\underline{g}^+$. We may assume that $\underline{g}$ is the maximal semi-simple ideal in $\underline{g}^+$.

Now let H^+ be defined by (7.5.1) so that there is a one-one relation between all dominant irreducible representations of H^+ (indexed by Λ) and all irreducible representations of G^+. Let $\delta \in \Lambda$ and let χ^δ be the character of the irreducible representation v^δ defined above. Let a $\in$ H^+ be regular in G^+ (every regular element of G^+ is conjugate to an element in H^+) and let W_a be the subgroup of W consisting of all $\sigma \in W$ which commute with $\tau(a)$ (see (7.5.2)). Then where $\chi_1^\sigma(a)$ and $\chi_1^{\sigma,\delta}(a)$ are given respectively by (7.5.4) and (7.5.6) one has

$$\chi^{\delta}(a) = \frac{\sum_{\sigma \in W_a} \operatorname{sg}\sigma \ \chi_1^{\sigma,\delta}(a)}{\sum_{\sigma \in W_a} \operatorname{sg}\sigma \ \chi_1^{\sigma}(a)}$$

Proof. Define $H(\underline{m},V^{\delta})$ with respect to the representation $\pi = \nu^{\delta}|\underline{m}$. By decomposing V^{δ} into irreducible components under the action of $\nu^{\delta}|\underline{g}$ it follows from Corollary 5.15 that the space of cochains $(e'_{-\Phi\sigma}) \otimes V_{\sigma}^{\delta}$ consists (except for zero) of non-cobounding cocycles and if $((e'_{-\Phi\sigma}) \otimes V_{\sigma}^{\delta})$ denotes the corresponding space of cohomology classes one has the direct sum

$$H(\underline{m},V^{\delta}) = \sum_{\sigma \in W} ((e'_{-\Phi\sigma}) \otimes V_{\sigma}^{\delta}) \ .$$

We have now only to apply Proposition 7.2 , (7.5.4) and (7.5.5). QED.

8. Application III.

Symmetric complex spaces X and a generalization of a theorem of Ehresmann. 1. Let $u \in U$ and let $\underline{g}_1$ and $\underline{n}$ be defined as in §5.3 . We continue with the notation of § 5 except that now it is assumed that $\lambda = 0$. Thus β is a representation of $\underline{g}_1$ on $\Lambda \underline{n}'$ and $\hat{\beta}$ is the induced representation of $\underline{g}_1$ on $H(\underline{n})$.

Now let β_* be the representation of $\underline{g}_1$ on $\Lambda \underline{n}$ defined by restricting $\theta|\underline{g}_1$ to $\Lambda \underline{n}$. Thus β_* is the representation contragredient to β. Since β_* obviously commutes with the

boundary operator ∂ on $\Lambda \underline{n}$ it defines a representation

$$\hat{\beta}_* : \underline{g} \longrightarrow \text{End } H_*(\underline{n})$$

on $\underline{g}_1$ on the homology group $H_*(\underline{n})$. It is of course clear that, with respect to the canonical duality between $H_*(\underline{n})$ and $H(\underline{n})$, $\hat{\beta}_*$ is just the representation contragredient to $\hat{\beta}$. Applying Theorem 5.14 one, then, immediately obtains

Corollary 8.1. *Let $u \in U$ and let $\hat{\beta}_*$ be the representation of $\underline{g}_1$ on the homology group $H_*(\underline{n})$ defined above. For any $\xi \in -D_1$ let $H_*(n)^{\xi}$ be the set of all elements in $H_*(n)$ which transform under $\hat{\beta}_*$ according to the irreducible representation (with lowest weight ξ) v_1^{ξ} of $\underline{g}_1$.*

Then for any $\sigma \in W^1$ one has $g - \sigma g \in -D_1$ and for any $\xi \in -D_1$ one has $H_*(n)^{\xi} \neq 0$ if and only if $\xi = g - \sigma g$ for some $\sigma \in W^1$. Furthermore $H_*(n)^{g-\sigma g}$ is irreducible for all $\sigma \in W^1$ so that $\sigma \longrightarrow H_*(n)^{g-\sigma g}$ is a bijection of W^1 onto the set of all irreducible (under $\hat{\beta}_*$) components of $H_*(n)$. Moreover, degree-wise, for any non-negative integer j,

$$H_j(\underline{n}) = \sum_{\sigma \in W^1(j)} H_*(\underline{n})^{g-\sigma g}$$

so that the elements of $H_*(\underline{n})^{g-\sigma g}$ are homogeneous of degree $n(\sigma)$. Finally the lowest weight vector of $H_*(\underline{n})^{g-\sigma g}$ if the homology class having $e_{\mathfrak{l}_\sigma}$ as a representative cycle.

8.2. We consider the cases ($u \in U$) when $\underline{n}$ is commutative. Let $\Pi(u) \subseteq \Pi$ be defined as in § 5.4 and for any $\varphi \in \Delta$ let the integer $n_\alpha(\varphi)$, $\alpha \in \Pi$, be defined also as in § 5.4.

It is then asserted that $\underline{n}$ is commutative if and only if for every

$$\sum_{\alpha \in \Pi(\underline{u})} n_\alpha(\varphi) \leq 1 \quad . \tag{8.2.1}$$

Indeed since $\Delta(\underline{n})$ is precisely the set of all $\varphi \in \Delta$ such that the left hand sum of (8.2.1) is ≥ 1 it follows that the condition (8.2.1) implies that n is commutative. On the other hand if there exists a root such that the left hand sum of (8.2.1) is ≥ 2 then since $\underline{m}$ is generated by the e_α, $\alpha \in \Pi$, it follows that there exists $\varphi \in \Delta(\underline{n})$ and $\alpha \in \Pi(\underline{u})$ such that $\varphi + \alpha \in \Delta$. But since φ, α, $\varphi + \alpha \in \Delta(\underline{n})$ this implies that $\underline{n}$ is not commutative. This proves the assertion. An immediate consequence of this and symmetric space theory is

Proposition 8.2. *Let $\underline{u} \in \underline{U}$ and, as in § 5.2, let $X = G/U$ so that X is a complex compact homogeneous space. Then X is also a symmetric space in the sense of E. Cartan if and only if $\underline{n}$ the maximal nilpotent ideal of $\underline{u}$, is commutative.*

Proof. It is immediate that the condition (8.2.1) is satisfied if and only if no two elements of $\Pi(\underline{u})$ lie in the same connected component (in the sense of Dynkin) of Π and for any $\alpha \in \Pi(\underline{u})$ one has $n_\alpha(\varphi) \leq 1$ for all $\varphi \in \Delta$. But then the result follows from the structure theory of complex, compact, symmetric spaces (see e.g.[3],40,p.260). QED.

But now if $\underline{n}$ is commutative the boundary operator on $\Lambda \underline{n}$ is zero. Thus $H(\underline{n}) = \Lambda \underline{n}$. Hence in the symmetric case

Corollary 8.1 yields Corollary 8.2 below describing how $\Lambda \underline{n}$ decomposes under the action of $\underline{g}_1$. Corollary 8.2 contains as a special case results of Ehresmann asserting how $\Lambda \underline{n}$ decomposes when X is symmetric and G is a classical group. We will work out the case when X is the Grassmannian in § 8.6.

Corollary 8.2. *Let $u \in U$. Assume that $X = G/U$ is a symmetric space. Let $\underline{n}$ be the maximal nilpotent ideal of $\underline{u}$ and let β_* be the representation of $\underline{g}_1$ on Λn obtained by restricting $\theta|\underline{g}_1$ to $\Lambda \underline{n}$. (Recall that θ is the adjoint representation of $\underline{g}$ on $\Lambda \underline{g}$).*

Now let W^1 be the subset of the Weyl group defined as in §5.13. Then for any $\sigma \in W^1$ one has $g - \sigma g \in -D_1$ and for any $\xi \in -D_1$ the irreducible representation ν_1^{ξ} of $\underline{g}_1$ occurs in β_* if and only if $\xi = g - \sigma g$ for some $\sigma \in W^1$. Furthermore if $\sigma \in W^1$ then $\nu_1^{g-\sigma g}$ occurs in β_* with multiplicity one and if $(\Lambda \underline{n})^{g-\sigma g}$ is the subspace on which it occurs (so that $\sigma \longrightarrow (\Lambda \underline{n})^{g-\sigma g}$ is a bijection of W^1 onto the set of all irreducible (under β_*) components of $\Lambda \underline{n}$) then the elements of $(\Lambda \underline{n})^{g-\sigma g}$ are all homogeneous $n(\sigma)$-vectors so that for any non-negative integer j one has the direct sum

$$\Lambda^j \underline{n} = \sum_{\sigma \in W^1(j)} (\Lambda \underline{n})^{g-\sigma g}$$

Finally the lowest weight vector in $(\Lambda \underline{n})^{g-\sigma g}$ is the decomposable $n(\sigma)$-vector e_{Φ_σ} (see § 5.11).

Remark 8.2. For usage in §8.6 it will be convenient to express Corollary 8.2 in terms of highest weights instead of lowest weights. In order to do this let $\mathcal{H} \in W$ be defined as in § 5.10 . Write (uniquely according to Proposition 5.13

$$\mathcal{H} = \mathcal{H}_1 \mathcal{H}^1$$

where $\mathcal{H}_1 \in W_1$ and $\mathcal{H}^1 \in W^1$. By (5.13.2) it follows that $\Phi_{\mathcal{H}_1} = \Delta(\mathfrak{m}_1)$ (see § 5.5). Thus for any $\xi \in -D_1$ one has that $\mathcal{H}_1(\xi) \in D_1$ and $\mathcal{H}_1(\xi)$ is the highest weight of v_1^{ξ}. It follows therefore that in the notation of § 5.5

$$(\Lambda \mathfrak{n})^{g - \sigma g} = (\Lambda \mathfrak{n})^{\mathcal{H}_1(g - \sigma g)}$$

and the highest weight in $\Lambda \mathfrak{n}^{g - \sigma g}$ is the decomposable $n(\sigma)$-vector $e_{\mathcal{H}_1(\Phi_\sigma)}$ where $\mathcal{H}_1(\Phi_\sigma)$ is the set of all roots (necessarily in $\Delta(\underline{n})$) of the form $\mathcal{H}_1 \varphi$ where $\varphi \in \Phi_\sigma$.

8.3. Let m be a positive integer and let $\underline{g}^m$ be the Lie algebra of all complex m × m matrices regarded as operating on $\underline{C}^m = V$ in the usual way. For any $y \in \underline{g}^m$ let y_{ij}, $i, j = 1, 2, \ldots, m$ be the matrix coefficients of y.

Now let $\underline{a}^m \subseteq \underline{g}^m$ be the set of m × m complex matrices of zero trace. We apply the considerations of § 8.2 to the case where $\underline{g} = \underline{a}^m$. We choose the maximal solvable Lie subalgebra $\underline{b}$ of $\underline{g}$ so that

$$\underline{b} = \{ y \in \underline{g} \mid y_{ij} = 0,\ i > j \}$$

(super-triangular matrices) and the Cartan subalgebra $\underline{h}$ of $\underline{g}$ so that $\underline{h}$ is the set of all diagonal matrices in $\underline{g}$. The

corresponding roots are then canonically indexed by all pairs $i,j = 1,2,\ldots,m$, $i \neq j$, where $\varphi_{ij} \in \Delta$ is given by

$$\langle \varphi_{ij}, x \rangle = x_{ii} - x_{jj} \tag{8.3.1}$$

for any $x \in \underline{h}$ and the corresponding root vectors may be chosen so that

$$e_{\varphi_{ij}} = \frac{1}{\sqrt{2m}} e_{ij}$$

where e_{ij} is the usual matrix unit.

Remark 8.3. The coefficient $\frac{1}{\sqrt{2m}}$ is necessary to insure the relation (5.1.2). It also insures (5.4.2) where $\underline{k}$ is chosen to be the set of all skew-Hermitian matrices in $\underline{g}$. Since $(\varphi, \varphi) = \frac{1}{m}$ for all $\varphi \in \Delta$ note also that we can write

$$\frac{\sqrt{2}}{|\varphi_{ij}|} e_{\varphi_{ij}} = e_{ij} \tag{8.3.2}$$

But using (5.1.3) this implies that

$$\frac{2}{(\varphi_{ij}, \varphi_{ij})} x_{\varphi_{ij}} = e_{ii} - e_{jj} \tag{8.3.3}$$

Now it is clear from the choice of $\underline{b}$ that $\Delta_+ = \Delta(\underline{b}) = \{\varphi_{ij}\}$, $i < j$. Hence the simple positive roots may be indexed so that $\Pi = \{\alpha_i\}$, $i = 1,2,\ldots,m-1$, where

$$\alpha_i = \varphi_{i,i+1} \quad .$$

Now let Z^m be the set of all m-tuples r,

$$r = (r_1, r_2, \ldots, r_m) \quad ,$$

where the r_i are integers. It is clear from § 5.5 and (8.3.3) that we can define a mapping

$$Z^m \longrightarrow Z, \quad r \longrightarrow \mu(r) \ ,$$

by letting $\mu(r)$ be defined by

$$\langle \mu(r), x \rangle = r_1 x_{11} + r_2 x_{22} + \ldots + r_m x_{mm}$$

for any $x \in \underline{h}$.

The Weyl group W may be identified with the permutation group on the numbers $1,2,\ldots,m$. If we let W operate on Z^m by putting

$$\sigma r = \{ r_{\sigma^{-1}(1)}, r_{\sigma^{-1}(2)}, \ldots, r_{\sigma^{-1}(m)}$$

then this identification may be made so that

$$\mu(\sigma r) = \sigma \mu(r)$$

for any $r \in Z^m$. Note then from (8.3.1) that for any root φ_{ij}

$$\sigma(\varphi_{ij}) = \varphi_{\sigma(i)\sigma(j)} \qquad (8.3.4)$$

Now let

$$D^m = \{ r \in Z \mid r_1 \geq r_2 \geq \ldots \geq r_m \} \ .$$

It is obvious that D^m is a fundamental domain for the action of W on Z^m and by (8.3.3) the mapping $r \longrightarrow \mu(r)$ carries D^m onto D.

Now for any $r \in Z^m$ let

$$n(r) = \sum_{i=1}^{m} r_i \ .$$

We recall some facts in the representation theory of $\underline{g}^m$. Let G^m be the group of all $m \times m$ complex non-singular matrices. For each $r \in Z^m$ let

$$\nu^r : \underline{g}^m \longrightarrow V^{\mu(r)}$$

be the irreducible representation of $\underline{g}^m$ on $V^{\mu(r)}$ defined so that $\nu^r \mid \underline{g} = \nu^{\mu(r)}$ and, if 1^m is the $m \times m$ identity matrix, $\nu^r(1^m)$ is the scalar $n(r)$ on $V^{\mu(r)}$.

Remark 8.3. Note that ν^{-r} is the contragredient representation to ν^r for any $r \in Z^m$.

It is clear that ν^r is equivalent to $\nu^{\sigma(r)}$ for any $\sigma \in W$. Also one knows that every irreducible representation of $\underline{g}^m$ which arises from an irreducible representation of G^m is equivalent to ν^r for one and only one element $r \in D^m$.

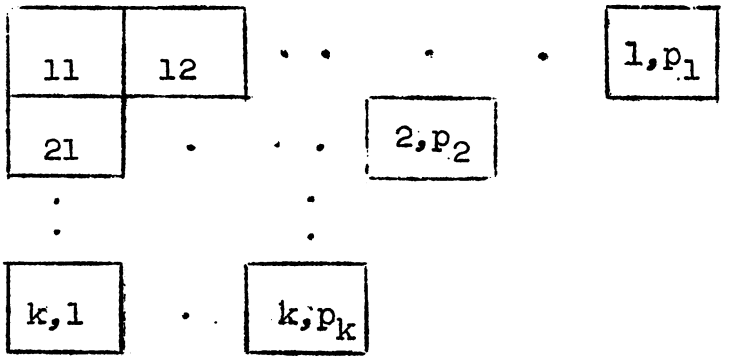
8.4. Now let P be the set of all partitions of all non-negative integers. Thus if $p \in P$ then p is given by a finite sequence

$$p = \{p_1, p_2, \ldots, p_k\}$$

where $p_1 \geq p_2 \geq \ldots \geq p_k$ are positive integers. To each $p \in P$ one associates two integers, $n(p)$, where

$$n(p) = \sum_{i=1}^{k} p_i$$

is the number being partitioned and $m(p)\,(= k)$, the number of parts. Also one associates to p a Young diagram $Y(p)$ which may be regarded as the set of all pairs (i,j) of positive integers where $1 \leq i \leq m(p)$ and $j \leq p_i$. Schematically the pairs of $Y(p)$ are represented by the boxes in the figure



(8.4.1)

Let $p \in P$. Then one associates with p another partition $\bar{p}$ (called its conjugate) where $m(\bar{p}) = p_1$ and $\bar{p}_j$, for $1 \leq j \leq p_1$, is given by

$$\bar{p}_j = \max_{(i,j) \in Y(p)} i \quad .$$

One has that $n(\bar{p}) = n(p)$, $m(\bar{p}) = p_1$, $m(p) = \bar{p}_1$ and schematically the box representation for $Y(\bar{p})$ is obtaining by transposing (8.4.1) as one would a matrix.

Now let

$$P^m = \{ p \in P \mid m(p) \leq m \}$$

and let

$$r^m : P^m \longrightarrow Z^m$$

be the mapping given by the relation $(r^m(p))_i = p_i$ for $1 \leq i \leq m(p)$ and $(r^m(p))_i = 0$ for $m(p) < i \leq m$. It is clear that r^m is a bijection of P^m onto the subset D^m_+ of D^m consisting of all $r \in D^m$ such that r_i is non-negative for all i.

Let $p \in P^m$. We recall (Young theory) how one obtains the representation $\nu^{r^m(p)}$ of $\underline{g}^m$. Let e_i, $i = 1,2,\ldots,m$, be a basis of V such that $e_{ii}(e_i) = e_i$ for all i. For any non-negative integer j let $\otimes^j V$ be the tensor product of V with itself j times and let ν^j be the representation of $\underline{g}^m$ on $\otimes^j V$ formed

by tensor product of the canonical representation of $\underline{g}^m$ on V with itself j times.

Now if $1 \leq j \leq m$ let $e_{(j)} \in \otimes^j V$ be alternating tensor defined by

$$e_{(j)} = \sum_{\sigma} sg\,\sigma\; e_{\sigma 1} \otimes \cdots \otimes e_{\sigma j}$$

where the sum is over all permutations on the numbers $1, 2, \ldots, j$.

Now let $e_{(p)} \in \otimes^{n(p)} V$ be defined by

$$e_{(p)} = e_{(\bar{p}_1)} \otimes e_{(\bar{p}_2)} \otimes \cdots \otimes e_{(\bar{p}_k)}$$

where $k = m(\bar{p}) = p_1$. Also let $V^p \subseteq \otimes^{n(p)} V$ be the subspace generated by $e_{(p)}$ under the representation $\nu^{n(p)}$. Then the subrepresentation of $\nu^{n(p)}$ defined by V^p is irreducible and is equivalent to $\nu^{r^m(p)}$. Moreover $e_{(p)}$ is a highest weight vector of $\nu^{n(p)}|_{\underline{g}}$.

Remark 8.4. One knows that for any integer j all the irreducible subrepresentations of ν^j are of the form ν^r where $r \in D_+^m$, that is, of the form $\nu^{r^m(p)}$ where $p \in P^m$. It follows therefore that, since this is true for all j, any irreducible subrepresentation of the tensor product $\nu^{r'} \otimes \nu^{r''}$ where $r', r'' \in D_+^m$ is again of the form ν^r where $r \in D_+^m$.

8.5. Let $\underline{U}$ be as in § 5.2. Let s be an integer where $1 \leq s \leq m$. We apply the considerations of § 8.2 to the case where X is the Grassmannian of all s complex planes in V. That is, to the case where $\underline{u} \in \underline{U}$ is the set of all matrices $y \in \underline{g}$ of the form

$$y = \begin{pmatrix} A_{11}(y) , & A_{12}(y) \\ 0 & A_{22}(y) \end{pmatrix}$$

where, if $t = m-s$, $A_{11}(y)$ is an $s \times s$ matrix, $A_{12}(y)$ is an $s \times t$ matrix and $A_{22}(y)$ is a $t \times t$ matrix.

It is clear that $\underline{n}$ is the set of all $y \in \underline{u}$ such that $A_{11}(y) = A_{22}(y) = 0$. Furthermore if k is chosen to be the set of all skew-Hermitian matrices in $\underline{g}$ then $\underline{g}_1$ is the set of all $y \in \underline{u}$ such that $A_{12}(y) = 0$. Also note that if $x \in \underline{g}_1$, $y \in \underline{n}$ and $z = [x,y] = \beta_*(x)y \in \underline{n}$ then

$$A_{12}(z) = A_{11}(x)A_{12}(y) - A_{12}(y)A_{22}(x) \qquad (8.5.1)$$

Now by definition (see (5.13.1)) W^1 is the set of all $\sigma \in W$ such that $\varphi \in \Delta_+$ and $\sigma^{-1}(\varphi) \in \Delta_-$ implies $\varphi \in \Delta(\underline{n})$. But now since $\Delta(\underline{n})$ is the set of all $\varphi_{ik} \in \Delta$ such that $1 \leq i \leq s$ and $s < k \leq m$ it is clear from (8.3.4) that $\sigma \in W^1$ if and only if

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(s)$$

and (8.5.2)

$$\sigma^{-1}(s+1) < \sigma^{-1}(s+2) < \ldots < \sigma^{-1}(m) \quad .$$

Remark 8.5. It follows immediately from (8.5.2) that an element $\sigma \in W^1$ is characterized by the values $\sigma^{-1}(i)$, $i = 1,2,\ldots,s$. Thus there are $\binom{m}{s}$ elements in W^1, the elements being in a canonical one-one correspondence with the set of all subsets of s integers between 1 and m.

Write $t = m-s$. Now let $\sigma \in W^1$. It is obvious from (8.5.2) that for any $1 \leq i \leq s$ and $1 \leq j \leq t$ one has

$$i \leq \sigma^{-1}(i) \quad \text{and} \quad j \leq \sigma^{-1}(s+j) \tag{8.5.3}$$

On the other hand we now observe that one or the other (but obviously not both) of the following inequalities must hold

$$\sigma^{-1}(i) < i+j \leq \sigma^{-1}(s+j)$$

or (8.5.4)

$$\sigma^{-1}(s+j) < i+j \leq \sigma^{-1}(i)$$

That is, in any event $i+j$ lies between $\sigma^{-1}(i)$ and $\sigma^{-1}(s+j)$ and is greater than the minimum of the two. This is an immediate consequence of (8.5.2) and the fact that σ^{-1} is a permutation.

We can now easily compute $\bar{\Phi}_\sigma$. By letting j vary from 1 to $\sigma^{-1}(i) - i$ it follows from (8.5.4) that

$$\bar{\Phi}_\sigma = \{\varphi_{i,s+j} \in \Delta \mid i \leq s, \quad i < \sigma^{-1}(i) \quad \text{and} \quad 1 \leq j \leq \sigma^{-1}(i) - i\} \tag{8.5.5}$$

Now given $\sigma \in W^1$ it follows immediately from (8.5.3) that we can define a partition $p^\sigma \in P^s$ by the relation

$$r^s(p^\sigma) = (\sigma^{-1}(s)-s, \ldots, \sigma^{-1}(1)-1) \tag{8.5.6}$$

But it is then an easy consequence of (8.5.4) that the conjugate partition $\overline{p^\sigma}$ lies in P^t and that

$$r^t(\overline{p^\sigma}) = (s+1 - \sigma^{-1}(s+1), \ldots, m- \sigma^{-1}(m)) \tag{8.5.7}$$

since the j^{th} entree in $r^t(p^{\overline{\sigma}})$ is by definition the number of entrees in $r^s(p^{\sigma})$ which $\geq j$.

Now let $q \in D^m$ be defined by

$$q = (m, m-1, \ldots, 1).$$

It is then clear from (5.9.1) and (8.3.3) that $\mu(q) = g$. On the other hand note that if $\sigma \in W$ then by

$$q - \sigma q = (\sigma^{-1}(1)-1, \ldots, \sigma^{-1}(m)-m)$$

Now we recall that the element $\mathcal{H}_1 \in W_1$ is characterized by the fact that $\Phi_{\mathcal{H}_1} = \Delta(m_1)$ (see Remark 8.2).

It follows easily then that

$$\mathcal{H}_1 = \mathcal{H}' \mathcal{H}'' \tag{8.5.8}$$

where

$$\mathcal{H}' = (1,s)(2,s-1), \ldots, \text{ and } \mathcal{H}'' = (s+1,m)(s+2,m-1)\ldots,$$

But then adjoining a t-tuple to an s-tuple to make an m-tuple it follows from (8.5.6) and (8.5.7) that for any $\sigma \in W^1$

$$\mathcal{H}_1(q - \sigma q) = (r^s(p^{\sigma}), -\mathcal{H}'' r^t(p^{\overline{\sigma}})) \tag{8.5.9}$$

On the other hand we clearly have that

$$\mu(\mathcal{H}_1(q - \sigma q)) = \mathcal{H}_1(g - \sigma g) \tag{8.5.10}$$

Also in terms of the Young diagram for p^{σ} note that we can write

$$\Phi_{\sigma} = \{\varphi_{i,s+j} \in \Delta \mid (\mathcal{H}'(i), j) \in Y(p^{\sigma})\}$$

so that by (8.5.8)

$$\mathcal{H}_1(\Phi_{\sigma}) = \{\varphi_{i,m+1-j} \in \Delta \mid (i,j) \in Y(p^{\sigma})\} \tag{8.5.11}$$

8.6. Now identify

$$\underline{g}^{s,t} = \underline{g}^{s} \oplus \underline{g}^{t}$$

with the set of all $m \times m$ matrices of the form y given in § 8.5 where $A_{12}(y) = 0$ and $A_{11}(y) \in \underline{g}^{s}$, $A_{21}(y) \in \underline{g}^{t}$ are arbitrary.

Let $r \in Z^{m}$. We may write, uniquely, $r = (r^{1}, r^{2})$ where $r^{1} \in Z^{s}$ and $r^{2} \in Z^{t}$. We will let ν_{1}^{r} be the irreducible representation of $G^{s,t} = G^{s} \times G^{t}$ on $V^{\mu(r^{1})} \otimes V^{\mu(r^{2})}$ given by

$$\nu_{1}^{(r^{1},r^{2})} = \nu^{r_{1}} \times \nu^{r_{2}} \quad .$$

It is clear then that every irreducible representation of $G^{s,t}$ is equivalent to ν_{1}^{r} for some $r \in Z^{m}$ and in fact r is unique chosen if one insists that $r^{1} \in D^{s}$ and $r^{2} \in D^{t}$.

Now the adjoint representation of $\underline{g}^{s,t}$ on $\underline{n}$ (see 8.5.1)) extends in the usual way to a representation

$$\beta_{s,t} : \underline{g}^{s,t} \longrightarrow \text{End } \Lambda \underline{n}$$

of $\underline{g}^{s,t}$ on Λn. We observe that the representation $\beta_{s,t}$ is obtained as an extension of the representation β_{*} of $\underline{g}_{1}$ on $\Lambda \underline{n}$ by defining $\beta_{s,t}(1^{m}) = 0$.

We wish to decompose the representation $\beta_{s,t}$ into irreducible components. We first observe, however, that if $r \in D^{t}$ then $-\mathcal{H}"r \in D^{t}$ and $\nu^{-\mathcal{H}"r}$ is equivalent to the representation contragredient to the representation ν^{r} of $\underline{g}^{t}$. This is clear from the definition of $\mathcal{H}"$. It follows easily therefore from (8.5.1) and Remark 8.4 that any irreducible

component of $\beta_{s,t}$ is of the form $\nu_1^{(r^1, -\mathcal{H}"r^2)}$ where $r^1 \in D_+^s$ and $r^2 \in D_+^t$.

Now let $Q^{s,t}$ be the set of all pairs (p^1,p^2) where $p^1 \in P^s$ and $p^2 \in P^t$ is such that $\nu_1^{(r^1,-\mathcal{H}"r^2)}$ occurs in the complete decomposition of $\beta_{s,t}$ if $r^1 = r^s(p^1)$ and $r^2 = r^t(p^2)$. From the remark above we see that every irreducible component of $\beta_{s,t}$ is of this form so that $\beta_{s,t}$ is determined as soon as the elements of $Q^{s,t}$ are known together with the corresponding multiplicities. The following theorem is due to Ehresmann.

Theorem 8.6. *Let $Q^{s,t}$ be the set of pairs of partitions defined above describing the decomposition of the representation $\beta_{s,t}$ of $\underline{g}^s \oplus \underline{g}^t$ ($\underline{g}^k$ is the Lie algebra of all $k \times k$ complex matrices) on $\Lambda \underline{n}$ where $\underline{n}$ is isomorphic to the space of all $s \times t$ complex matrices.*

Let W^1 be defined as in §5.13 so that here W^1 is the set of permutations $\sigma \in W$ satisfying (8.5.2). If $\sigma \in W^1$ let $p^\sigma \in P^s$ be the partition defined by

$$r^s(p^\sigma) = (\sigma^{-1}(s)-s, \ldots, \sigma^{-1}(1)-1).$$

Then

$$n(p^\sigma) = n(\sigma) \tag{8.6.1}$$

where the left and right sides of (8.6.1) are defined respectively as in § 8.4 and by (5.13.3). Furthermore $\sigma \longrightarrow p^\sigma$ is a bijection of W^1 onto the set of all partitions p such that

$$m(p) \leq s \text{ and } m(\overline{p}) \leq t$$

where $\bar{p}$ is the conjugate partition (that is, the set of all partitions whose Young diagram (block representation) "fits" into an s × t rectangle of blocks).

Finally $Q^{s,t}$ is the set of all pairs $(p^\sigma, p^{\bar{\sigma}})$ where σ runs through W^1. Moreover the irreducible representation of $g^{s,t}$ corresponding to any pair $(p^\sigma, p^{\bar{\sigma}})$ occurs with multiplicity one and the representation induced on g_1 is $\nu_1^{\mathcal{H}_1(g-\sigma g)}$. Moreover the space of the representation consists of homogeneous $n(p^\sigma)$ vectors and a highest weight vector of $\nu^{\mathcal{H}_1(g-\sigma g)}$ is (in any order)

$$\overline{\prod}_{(i,j) \in Y(p^\sigma)} e_{i,m+1-j}$$

where $\overline{\prod}$ denotes exterior multiplication.

Proof. The equality $n(p^\sigma) = n(\sigma)$ follows from (8.5.11) and the other statements about p^σ follow from Remark 8.5. To prove the theorem therefore we have only to apply Corollary 8.2 and Remark 8.2 and to determine the element of $Q^{s,t}$ corresponding to representation $\nu_1^{\mathcal{H}_1(g-\sigma g)}$ of g_1 on the subspace $(\bigwedge \underline{n})^{\mathcal{H}_1(g-\sigma g)}$ of $\bigwedge^{n(\sigma)} \underline{n}$. That is we must find the pair $(p^1, p^2) \in Q^{s,t}$ such that (1)

$$\mu(r^s(p^1), -\mathcal{H}"r^t(p^2)) = \mathcal{H}_1(g-\sigma g)$$

and (2) $n(p^1) = n(\sigma)$ (since $\beta_{s,t}(y)$ must reduce to the scalar $n(\sigma)$ on $\bigwedge^{n(\sigma)} \underline{n}$ if $y \in g^{s,t}$ is the element such that $A_{22}(y) = 0$ and $A_{11}(y) = 1^s$. It is easy to see that (1) and (2)

define (p^1,p^2) uniquely. But by (8.5.9), (8.9.10), and the equality (8.6.1), it follows that $(p^1,p^2) = (p^\sigma,\overline{p^\sigma})$. The final statement follows from (8.5.11) and Remark 8.2. QED.

Remark 8.6. Theorem 8.6 lends some insight into the nature of the weight $g-\sigma g$, at least for the case at hand. The striking thing is that the partitions p^1 and p^2 of the pair (p^1,p^2) corresponding to the weight $g-\sigma g$ not only determine each other but are related to the extent that one is the conjugate of the other. Furthermore except for a limitation on size the choice of p^1 can be made arbitrary by choosing σ properly.

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Berkeley

ABSTRACT

This report is Part I of a paper consisting of two parts. The results of this paper include a extension of the Schubert calculus and the solution to a half-century old problem of relating the Schubert calculus with the multiplication of Frobenius S-functions. In Part I we are concerned mainly with the determination of the cohomology groups associated with a familiar family of nilpotent Lie algebras. The main results in this direction are Theorems 4.4, 5.7 and 5.14. Applications are given in §§6, 7 and 8. Among the applications is a proof of the generalized Borel-Weil theorem. This theorem was conjectured by Borel and Hirzebruch. It was first proved by R. Bott. His proof, however, is extremely complicated and uses very deep results in algebraic geometry. The proof given here is much more direct and has the additional advantage of using methods which have other applications, e.g. to the extended Schubert calculus mentioned above.